Index formula for families of end-periodic Dirac operators

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University of Illinois at Urbana-Champaign

Geometric Analysis Seminar

1 Index theory for Dirac operators

2 Index theory for end-periodic Dirac operators

3 A new index formula for families

Geometric Dirac operators

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satisfying $\mathrm{c}\ell(e^i)\,\mathrm{c}\ell(e^j) + \mathrm{c}\ell(e^j)\,\mathrm{c}\ell(e^i) = -2g^{ij}\,\mathrm{id}$
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ex. Gauss-Bonnet operator: $D^+ = d + d^*$ acting on $\Omega^{\mathrm{ev}}(M) \subseteq \Omega^\bullet(M)$

ex. Signature operator: $D^+ = d + d^*$ acting on self-dual forms ($*\omega = \omega$).

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ex. $\operatorname{ind}(d + d^*) = \chi(M) := \sum_{k=0}^{n} (-1)^k \dim H^k_{dR}(M)$.

Consider simple situation where $\dim M = 2$. Gauss-Bonnet theorem tells us that

$$\operatorname{ind}(d + d^*) = \chi(M) = \frac{1}{2\pi}\int_M K\,dA.$$

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Notation: $AS(D(M)) = \dfrac{1}{(2\pi i)^{n/2}} \widehat{A}(TM) \wedge \mathrm{ch}'(E)$.

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Define the supertrace of the heat operator

$$\mathrm{Str}(e^{-tD^2}) := \mathrm{Tr}(e^{-tD^-D^+}) - \mathrm{Tr}(e^{-tD^+D^-}).$$

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\end{aligned}$$

End-periodic manifolds

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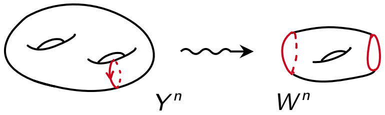
(Taubes 1987)



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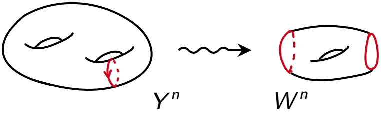

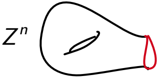

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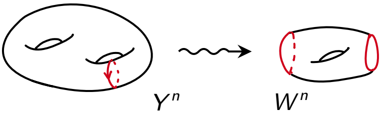

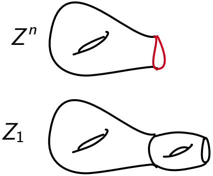

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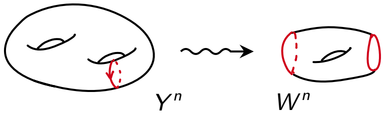

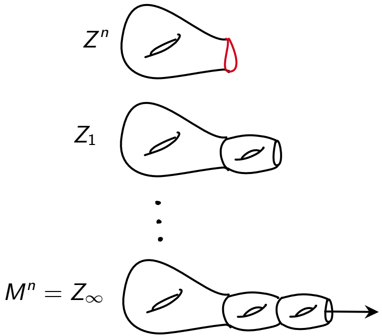

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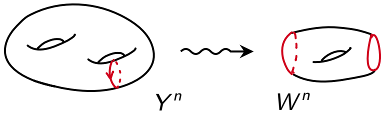

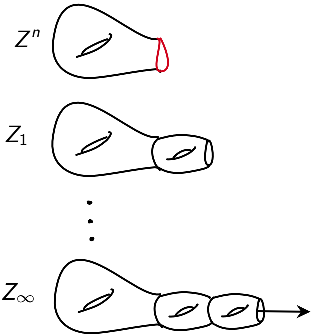

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\begin{tikzcd}
\widetilde{Y} \arrow[r, "f"] \arrow[d, "p"] & \mathbb{R} \arrow[d, "\exp"] \\
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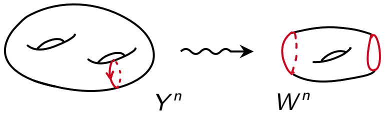

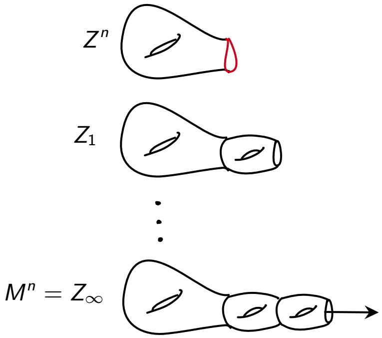

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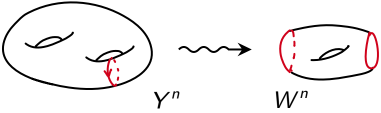

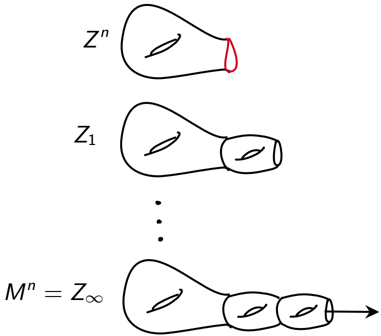

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Family context :
$M \to B,\ Y \to B,\ \widetilde{Y} \to B$
$\mathbb{E} = E \otimes \pi^*\Lambda T^*B,\ etc.$

Replace :
$E_Y \rightsquigarrow \mathbb{E},\ \widetilde{E} \rightsquigarrow \widetilde{\mathbb{E}}$
$E \rightsquigarrow \mathbb{E},\ D \rightsquigarrow A,\ etc.$

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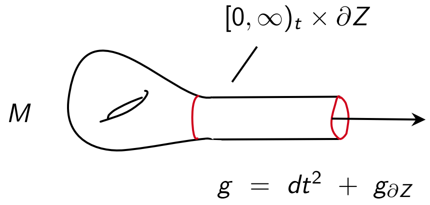

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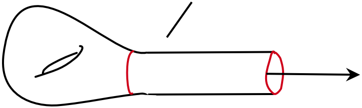

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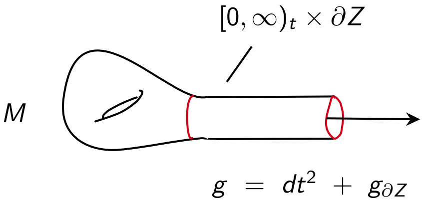

Other examples:

(i) $g = dt^2 + h(t)^2 g_{\partial Z}$ on the cylinder

(ii) $g = u^{4/(n-2)}(dt^2 + d\varphi^2)$ on $\mathbb{R} \times S^{n-1}$ (Mazzeo, Pollack, Uhlenbeck, 1994)

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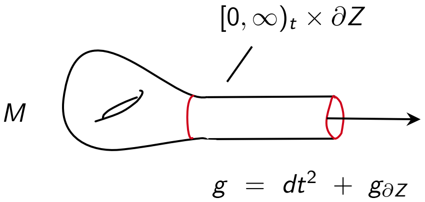

Other examples:

(i) $g = dt^2 + h(t)^2 g_{\partial Z}$ on the cylinder

(ii) $g = u^{4/(n-2)}(dt^2 + d\varphi^2)$ on $\mathbb{R} \times S^{n-1}$ (Mazzeo, Pollack, Uhlenbeck, 1994)

(iii) "Manifolds with periodic ends that are not products even topologically ... include manifolds whose ends arise from the infinite cyclic covers of 2-knot exteriors in the 4-sphere." (Mrowka, Ruberman, Saveliev, 2014)

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$$ {}^R\mathrm{Tr}\, P = \lim_{N\to\infty} \left[\int_{Z_N} \mathrm{tr}(K_P(x,x))\, dx - (N+1) \int_W \mathrm{tr}(K_{\widetilde{P}}(x,x))\, dx \right]. $$

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For any $\xi \in S^1$, define $\mathcal{F}_\xi : C_c^\infty(\widetilde{Y}; \widetilde{E}) \to C^\infty(Y; E_Y)$ by

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Periodic + family + $\mathbb{Z}_2$-graded (T. 2025)

$$ \begin{aligned} {}^R\mathrm{Str}[P,Q] = &\frac{-1}{2\pi i} \oint_{S^1} \mathrm{Str}\left(\frac{\partial P_\xi}{\partial \xi} Q_\xi\right) d\xi \\ &+ \frac{1}{2\pi i} \oint_{S^1} \int_{W/B} f(x)\, \mathrm{str}\left(K_{[P_\xi,Q_\xi]}(x,x)\right) dx\, \frac{d\xi}{\xi}. \end{aligned} $$

Atiyah-Patodi-Singer & MRS index formulas

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Families of Dirac operators

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M_z — $M \xrightarrow{\pi} B$ Riemannian fiber bundle, with $TM = \pi^* TB \oplus T(M/B)$.
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$$\begin{aligned} &\operatorname{ch} : K^0(B) \to H^{\mathrm{even}}_{dR}(B) \\ &\operatorname{ch}([E] - [F]) = [\operatorname{ch}(\nabla^E)] - [\operatorname{ch}(\nabla^F)]. \end{aligned}$$

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Goal: find a nice representative for $\mathrm{ch}(\operatorname{Ind} D)$ in de Rham cohomology.

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$$A = \sum_j m_0(e^j)\nabla^{\mathbb{E},0}_{e_j}$$

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End-periodic case: also have $A(Y)$ on $\mathbb{E}_Y$, and $\widetilde{A}$ on $\widetilde{\mathbb{E}}$...

EP & cylindrical family index formula: step 1

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Melrose & Piazza first step: $\dfrac{d}{dt}\,{}^b\mathrm{ch}(A_t) = -\,{}^b\mathrm{Str}[A_t, \dot{A}_t e^{-A_t^2}]$.

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Proof. By Duhamel's formula, the derivative of the heat kernel is

$$\frac{d}{dt}e^{-\mathbb{A}_t^2} = -\int_0^1 e^{-s\mathbb{A}_t^2}\cdot\frac{d\mathbb{A}_t^2}{dt}\cdot e^{-(1-s)\mathbb{A}_t^2}\,ds. \tag{13.10}$$

^{b}Str defect formula

* The indicial family of $\mathbb{A}_t^2$ is even in λ, so applying (12.5) to commute the leading term, from left to right, no boundary terms arise. Thus

$$\frac{d}{dt}\,b\text{-Ch}(\mathbb{A}_t) = -\,b\text{-STr}\left(\frac{d\mathbb{A}_t^2}{dt}e^{-\mathbb{A}_t^2}\right). \tag{13.11}$$

Since $\mathbb{A}_t$ is odd and commutes with $\exp(-\mathbb{A}_t^2)$, this can be written as a supercommutator:

$$\frac{d}{dt}\,b\text{-Ch}(\mathbb{A}_t) = -\,b\text{-STr}[\mathbb{A}_t, \frac{d\mathbb{A}_t}{dt}e^{-\mathbb{A}_t^2}]. \tag{13.12}$$

Now, $d\mathbb{A}_t/dt$ is a fibre operator so (12.5) can be applied to give

Complication in the end-periodic case

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Take $P = e^{-sA_t^2}$ and $Q = \partial_t(A_t^2)e^{-(1-s)A_t^2}$ in the supertrace defect formula:

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{}^R\mathrm{Str}[e^{-sA_t^2}, \partial_t(A_t^2)e^{-(1-s)A_t^2}] = \alpha_1(t) + \alpha_2(t).
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$$A_t(\xi) = A_t(Y) - \ln(\xi)\delta_t m_0(df)$$

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Proposition (T., 2025)

$$\begin{aligned}[A(Y), m_0(df)] &= m_0\left(\nabla^{T^*Y} df\right) - \nabla^{\mathbb{E}_Y,0}_{gradf} \\ &= 0 \quad \text{iff the periodic end is cylindrical}\end{aligned}$$

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$$e^{-A_t^2(\xi)} = ???$$

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$$= -\frac{1}{2}\widehat{\eta}(t) - (\text{exact})$$

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EP & cylindrical family index formula: step 2

End-periodic

$$\frac{d}{dt}\,{}^{R}\mathrm{ch}(A_t) = -\,{}^{R}\mathrm{Str}\left[A_t, \dot{A}_t e^{-A_t^2}\right] + \alpha_1(t) + \alpha_2(t)$$

$$= \beta_1(t) + \beta_2(t) + \alpha_1(t) + \alpha_2(t) + (\text{exact})$$

Cylindrical

$$\frac{d}{dt}\,{}^{b}\mathrm{ch}(A_t) = -\,{}^{b}\mathrm{Str}\left[A_t, \dot{A}_t e^{-A_t^2}\right]$$

$$= -\,{}^{b}\mathrm{Str}\left[A_t - A_{[1]}, \dot{A}_t e^{-A_t^2}\right] - {}^{b}\mathrm{Str}\left[A_{[1]}, \dot{A}_t e^{-A_t^2}\right]$$

$$= -\frac{1}{2}\widehat{\eta}(t) - (\text{exact})$$

$$= -\frac{1}{2}\widehat{\eta}_{\mathrm{ep}}(t) - \frac{d}{dt}\oint_{S^1}\int_{W/B} f(x)\,\mathrm{str}(K_{e^{-A_t^2(\xi)}}(x,x))\,dx\,\frac{d\xi}{\xi} + (\text{exact})$$

$$(*)\quad -\frac{1}{2}\widehat{\eta}_{\mathrm{ep}}(t) = \alpha_1(t) + \beta_1(t) = \alpha_1(t) - \frac{1}{2\pi i}\oint_{|\xi|=1} t^{1/2}\,\mathrm{Str}\left(c\ell(d_{Y/B}f)\dot{A}_t(Y)(\xi)e^{-A_t^2(Y)(\xi)}\right)\frac{d\xi}{\xi}$$

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$$-\frac{1}{2}\widehat{\eta}_{\mathrm{ep}}(t) - d_B\left(\frac{1}{2\pi i}\oint_{|\xi|=1}\mathrm{Str}\left(\frac{\partial\mathcal{Q}_\xi}{\partial\xi}\right)\,d\xi\right).$$

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The end-periodic eta form is given by

$$\frac{1}{2}\widehat{\eta}_{\mathrm{ep}}(t) = \frac{-1}{2\pi i}\oint_{|\xi|=1} t^{1/2}\,\mathrm{Str}\left(\mathrm{c}\ell(d_{Y/B}f)\dot{A}_t(Y)(\xi)e^{-A_t^2(Y)(\xi)}\right)\frac{d\xi}{\xi}$$
$$+\frac{1}{2\pi i}\oint_{|\xi|=1}\mathrm{Str}\left(\frac{d}{dt}[A_t(Y)(\xi),\delta_t m_0(df)]\mathcal{H}_t(\xi)\right)\frac{d\xi}{\xi}$$

with $\mathcal{H}_t(\xi) = \int_0^1 e^{-uA_t^2(\xi)}\,du.$

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$$\widehat{\eta}_{\mathrm{ep}} = \int_0^\infty \widehat{\eta}_{\mathrm{ep}}(t)\, dt.$$

The degree 0 component of $\widehat{\eta}_{\mathrm{ep}}$ is the fiberwise end-periodic eta invariant $z \mapsto \eta_{\mathrm{ep}}(D^z(Y_z))$.

Thank you

A few references:

1. C. Taubes, *Gauge theory on end-periodic 4-manifolds*, 1987.
2. N. Berline, E. Getzler, M. Vergne, *Heat Kernels and Dirac Operators*, 1992.
3. R. Melrose, P. Piazza, *Families of Dirac Operators, Boundaries, and the b-Calculus*, 1997.
4. T. Mrowka, D. Ruberman, N. Saveliev, *An index theorem for end-periodic operators*, 2014.

Extra slide

Some other important operators:

$$\delta_t m_0(df) = t^{1/2}\,\mathrm{c}\ell(d_{Y/B} f) + \pi^* d_B f$$

$$A_t(Y)(\xi) = A_t(Y) - \log(\xi)\delta_t m_0(df)$$

$$\mathcal{Q}_\xi = \dot{A}_t(Y)(\xi) e^{-A_t^2(Y)(\xi)}$$

The end-periodic eta invariant in the MRS index formula is given by

$$\frac{1}{2}\widehat{\eta}_{\mathrm{ep}} = \frac{1}{2\pi i}\int_0^\infty \oint_{|\xi|=1} \mathrm{Tr}(\mathrm{c}\ell(df) D_\xi^+ \exp(-t D_\xi^- D_\xi^+)\,\frac{d\xi}{\xi}\,dt.$$

Extra slide

The Bismut superconnection on Y is

$$A(Y) = D(Y) + A_{[1]}(Y) + A_{[2]}(Y)$$

where

$$A_{[1]}(Y) = \sum_{\alpha} e^{\alpha} \wedge \left(\nabla^{E_Y}_{e_\alpha} + \frac{1}{2} k_Y(e_\alpha) \right)$$

$$A_{[2]}(Y) = -\frac{1}{4} \sum_{\alpha<\beta} \sum_{j} e^{\alpha} \wedge e^{\beta} \, c\ell(e^{j}) \Omega_Y(e_\alpha, e_\beta) e_j$$