

Families index for end-periodic Dirac operators and gravitational anomalies in QFT

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- 1 Index theory for Dirac operators
- 2 Index theory and anomalies
- 3 Index theory for end-periodic Dirac operators
- 4 A new index formula
- 5 Remarks on the proof depending on how much time is left

Geometric Dirac operators

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satisfying $\text{cl}(e^i) \text{cl}(e^j) + \text{cl}(e^j) \text{cl}(e^i) = -2g^{ij} \text{id}$

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ex. $D^+ = d + d^*$ acting on self-dual forms $\Omega_+^{n/2}(M)$, i.e. $*\omega = \omega$
(signature operator).

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ex. $\operatorname{ind}(d + d^*) = \chi(M) := \sum_{k=0}^n (-1)^k \dim H_{dR}^k(M).$

Consider simple situation where $\dim M = 2$. Gauss-Bonnet theorem tells us that

$$\operatorname{ind}(d + d^*) = \chi(M) = \frac{1}{2\pi} \int_M K \, dA.$$

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Notation: $\text{ind } D^+ = \int_M AS(D(M)).$

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where the *eta invariant* is defined as

$$\eta(D(\partial Z)) = \frac{1}{\sqrt{\pi}} \int_0^\infty t^{-1/2} \operatorname{Tr}(D(\partial Z) e^{-tD(\partial Z)^2}) dt.$$

Anomalies in QFT

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(M, g_0) smooth Riemannian spin manifold. $\pi : M \rightarrow M$ a diffeomorphism. Given a metric g , Witten considers the *effective action* for a Majorana-Weyl fermion:

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Consider the mapping torus $(M \times S^1)_\pi = (M \times [0, 1]) / \sim$, where $(x, 1) \sim (\pi(x), 0)$. Suppose $(M \times S^1)_\pi = \partial B$. Then using APS, Witten computes ΔI in terms of the eta invariant $\eta(D(\partial B))$.

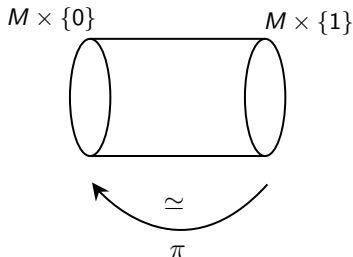
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Anomalies in QFT

Global gravitational anomalies, Witten, 1985

where $H = dB + \omega_3^L$. Thus, the non-covariant objects dB and ω combine, as they must, into the covariant field strength H . ($\Delta\tilde{S}$ and ΔI_{Reg} were invariant under coordinate transformations that vanish at $t=0$ and $t=1$, but not otherwise. As far as I know, it does not make sense to identify $t=1$ with $t=0$ by means of a non-trivial diffeomorphism π until these expressions are combined into a covariant form.) Thus we finally get an expression for the change in the action in terms of topological invariants:

$$\begin{aligned}\Delta I_{\text{TOT}} &= \Delta I_{\text{det}} + \Delta\tilde{S} + \Delta I_{\text{Reg}} \\ &= -2\pi i \left[\frac{1}{(2\pi)^6} \int_B \left(\frac{1}{1536} (\text{Tr } R^2)^3 + \frac{1}{384} \text{Tr } R^2 \text{Tr } R^4 \right) \right. \\ &\quad \left. - \int_{(M \times S^1)_\pi} H \cdot \left(\frac{1}{1536} (\text{Tr } R^2)^2 + \frac{1}{384} \text{Tr } R^4 \right) \right] \\ &= -2\pi i \cdot \frac{1}{192} [-3p_1^3(B) + 4p_1 p_2(B)],\end{aligned}\tag{57}$$

where p_1^3 and $p_1 p_2$ were defined in Sect. II.

Families of Dirac operators

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$D = (D^z)_{z \in B}$ = family of Dirac operators on family of Clifford modules $E_z \rightarrow M_z$.
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Important construction: the **Bismut superconnection**

$$A \circ C^\infty(M; E \otimes \wedge T^*B)$$

$$A = D + A_{[1]} + A_{[2]}.$$

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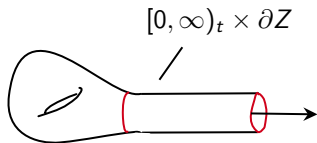
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A path of metrics g_t is a loop in the parameter space B and Witten's global anomaly is recovered as the holonomy around this loop of the connection on the determinant line bundle.

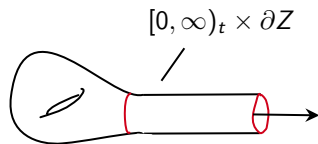
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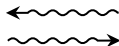


$$g = dt^2 + g_{\partial Z}$$

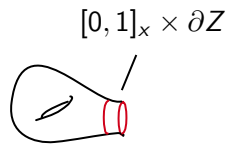
Cylindrical end = boundary



$$g = dt^2 + g_{\partial Z}$$



$$x = e^{-t}$$



$$g = \frac{dx^2}{x^2} + g_{\partial Z}$$

End-periodic manifolds

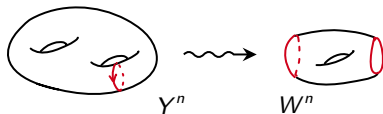
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(Taubes 1987)



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$$\begin{array}{ccc}
 \tilde{Y} & \xrightarrow{f} & \mathbb{R} \\
 \downarrow p & & \downarrow \exp \\
 Y & \xrightarrow{\bar{f}} & S^1
 \end{array}$$

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 E|_{\text{end}(M)} & \xrightarrow{\quad} & \text{end}(M) & & \\
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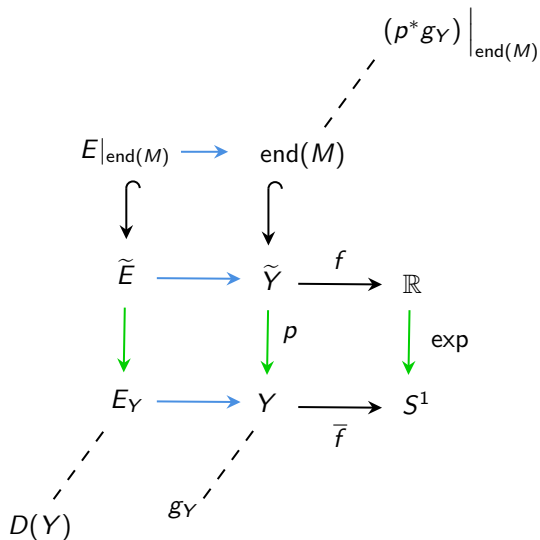
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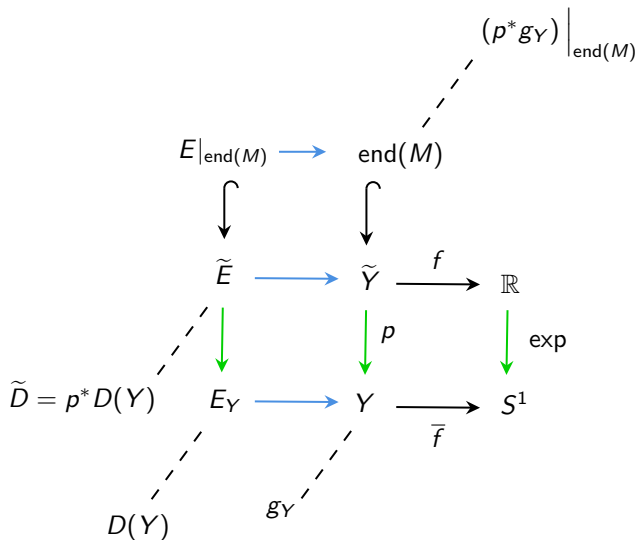
End-periodic Dirac operators

$$\begin{array}{ccccc}
 & & & & (p^* g_Y) \Big|_{\text{end}(M)} \\
 & & & \nearrow \text{dashed} & \\
 E|_{\text{end}(M)} & \xrightarrow{\quad} & \text{end}(M) & & \\
 \downarrow \curvearrowright & & \downarrow \curvearrowright & & \\
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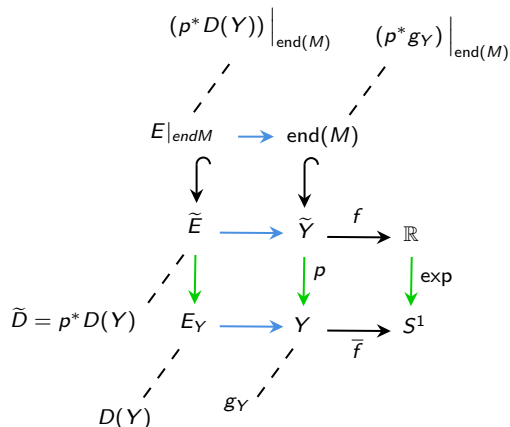
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 & \swarrow \text{dashed} & & \swarrow \text{dashed} & \\
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 D(Y) & g_Y & & &
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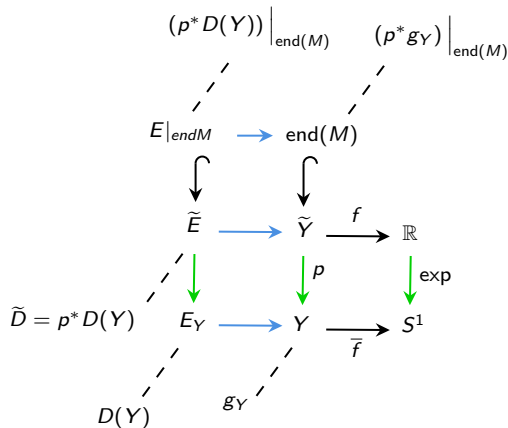


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Family context :

$$M \rightarrow B, Y \rightarrow B, \tilde{Y} \rightarrow B$$

$$\mathbb{E} = E \otimes \pi^* \Lambda T^* B, \text{ etc.}$$

Replace :

$$E_Y \rightsquigarrow \mathbb{E}, \tilde{E} \rightsquigarrow \tilde{\mathbb{E}}$$

$$E \rightsquigarrow \mathbb{E}, D \rightsquigarrow A, \text{ etc.}$$

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- “Manifolds with periodic ends that are not products even topologically ... include manifolds whose ends arise from the infinite cyclic covers of 2-knot exteriors in the 4-sphere.” (Mrowka, Ruberman, Saveliev, 2014).

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- *Metrics on end-periodic manifolds as models for dark matter*, Duston, 2021.

Index formula for end-periodic Dirac operators

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Mrowka, Ruberman, Saveliev, 2014: under the appropriate Fredholm condition, the index of the end-periodic Dirac operator D is given by

$$\operatorname{ind} D^+ = \int_Z AS(D(Z)) - \int_Y f AS(D(Y)) - \frac{1}{2} \eta_{\text{ep}}(D(Y))$$

where the end-periodic eta invariant is defined by

$$\widehat{\eta}_{\text{ep}}(D(Y)) = \frac{1}{\pi i} \int_0^\infty \oint_{S^1} \operatorname{Tr} \left(\operatorname{cl}(df) D_\xi^+ e^{-t D_\xi^- D_\xi^+} \right) \frac{d\xi}{\xi} dt$$

and $D_\xi = D(Y) - \ln(\xi) \operatorname{cl}(df)$, for $\xi \in S^1$, is the *indicial family* of D .

- 1 Index theory for Dirac operators
- 2 Index theory and anomalies
- 3 Index theory for end-periodic Dirac operators
- 4 A new index formula**
- 5 Remarks on the proof depending on how much time is left

Index for families of end-periodic Dirac operators

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Theorem (T.) *The Chern character of the index bundle, $\text{ch}(\text{Ind } D)$, is represented in de Rham cohomology by the differential form*

$$\int_{Z/B} AS(D(Z)) - \int_{W_0/B} f AS(D(Y)) - \frac{1}{2} \widehat{\eta}_{\text{ep}}(D(Y)) \in \Omega(B)$$

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where $\mathcal{H}_t(\xi) = \int_0^1 e^{-uA_t^2(\xi)} du$.

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where $\mathcal{H}_t(\xi) = \int_0^1 e^{-uA_t^2(\xi)} du$. Moreover, the degree zero part of the end-periodic eta form $\hat{\eta}_{\text{ep}}$ is the fiberwise end-periodic eta invariant: $(\hat{\eta}_{\text{ep}})_{[0]} \in C^\infty(B)$ given by $z \mapsto \eta_{\text{ep}}(D^z(Y_z))$.

Thank you

Extra slide

Some other important operators:

$$m_0 : T^*M \rightarrow \text{End } \mathbb{E}$$

$$\delta_t m_0(df) = t^{1/2} \text{cl}(d_{Y/B}f) + \pi^* d_B f$$

$$A_t(Y)(\xi) = A_t(Y) - \log(\xi) \delta_t m_0(df)$$

Extra slide

The Bismut superconnection on Y is

$$A(Y) = D(Y) + A_{[1]}(Y) + A_{[2]}(Y)$$

where

$$A_{[1]}(Y) = \sum_{\alpha} e^{\alpha} \wedge \left(\nabla_{e_{\alpha}}^{E_Y} + \frac{1}{2} k_Y(e_{\alpha}) \right)$$
$$A_{[2]}(Y) = -\frac{1}{4} \sum_{\alpha < \beta} \sum_j e^{\alpha} \wedge e^{\beta} \operatorname{cl}(e^j)(\Omega_Y(e_{\alpha}, e_{\beta}), e_j)$$

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$$\mathrm{Tr}(e^{-tD^- D^+}) = \sum_j \langle e^{-tD^- D^+} \psi_j^+, \psi_j^+ \rangle$$

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Define the supertrace of the heat operator

$$\mathrm{Str}(e^{-tD^2}) := \mathrm{Tr}(e^{-tD^- D^+}) - \mathrm{Tr}(e^{-tD^+ D^-}).$$

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Renormalized trace defect

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Cylindrical (Melrose)

$${}^b\mathrm{Tr}[P, Q] = \frac{-1}{2\pi i} \oint_{S^1} \mathrm{Tr} \left(\frac{\partial P_\xi}{\partial \xi} Q_\xi \right) d\xi.$$

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Periodic + family + \mathbb{Z}_2 -graded (T. 2025)

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EP & cylindrical family transgression formulas

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End-periodic

Cylindrical

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$$\begin{aligned}\frac{d}{dt} {}^R\text{ch}(A_t) &= - {}^R\text{Str} \left[A_t, \dot{A}_t e^{-A_t^2} \right] \\ &\quad + \alpha_1(t) + \alpha_2(t)\end{aligned}$$

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$$(*) \quad -\frac{1}{2} \widehat{\eta}_{\text{ep}}(t) = \alpha_1(t) + \beta_1(t)$$

$$= \alpha_1(t) - \frac{1}{2\pi i} \oint_{|\xi|=1} t^{1/2} \text{Str} \left(\text{cl}(d_{Y/B} f) \dot{A}_t(Y)(\xi) e^{-A_t^2(Y)(\xi)} \right) \frac{d\xi}{\xi}$$

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EP & cylindrical family transgression formulas

Melrose-Piazza (1997) first step: $\frac{d}{dt} {}^b\text{ch}(A_t) = - {}^b\text{Str}[A_t, \dot{A}_t e^{-A_t^2}]$.

Proof. By Duhamel's formula, the derivative of the heat kernel is

$$(13.10) \quad \frac{d}{dt} e^{-A_t^2} = - \int_0^1 e^{-sA_t^2} \cdot \frac{dA_t^2}{dt} \cdot e^{-(1-s)A_t^2} ds.$$

bStr defect formula

A (The indicial family of A_t^2 is even in λ , so applying (12.5) to commute the leading term, from left to right, no boundary terms arise. Thus

$$(13.11) \quad \frac{d}{dt} {}^b\text{Ch}(A_t) = - {}^b\text{STr} \left(\frac{dA_t^2}{dt} e^{-A_t^2} \right).$$

Since A_t is odd and commutes with $\exp(-A_t^2)$, this can be written as a supercommutator:

$$(13.12) \quad \frac{d}{dt} {}^b\text{Ch}(A_t) = - {}^b\text{STr}[A_t, \frac{dA_t}{dt} e^{-A_t^2}].$$

Now, dA_t/dt is a fibre operator so (12.5) can be applied to give

Complication in the end-periodic case

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$$A_t(\xi) = A_t(Y) - \ln(\xi)\delta_t m_0(df)$$

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Proposition (T., 2025)

$$\begin{aligned} [A(Y), m_0(df)] &= m_0 \left(\nabla^{T^*Y} df \right) - 2 \nabla_{(d_{Y/B} f)^\#}^{\mathbb{E}_{Y,0}} \\ &= 0 \quad \text{iff the periodic end is cylindrical} \end{aligned}$$

Complication in the end-periodic case

End-periodic

$$A_t(\xi) = A_t(Y) - \ln(\xi)\delta_t m_0(df)$$

$$A_t^2(\xi) = A_t^2(Y) - \ln(\xi)\delta_t[A(Y), m_0(df)]$$

$$- t \ln(\xi)^2 |d_{Y/B} f|^2$$

Cylindrical

$$A_t(\xi) = A_t(Y) - \ln(\xi)\delta_t m_0(dr)$$

$$A_t^2(\xi) = A_t^2(Y) - \ln(\xi)\delta_t[A(Y), m_0(dr)]$$

$$- t \ln(\xi)^2 |d_{Y/B} r|^2$$

$$= A_t^2(Y) - t \ln(\xi)^2$$

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$$A_t(\xi) = A_t(Y) - \ln(\xi) \delta_t m_0(df)$$

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$$- t \ln(\xi)^2 |d_{Y/B} f|^2$$

$$e^{-A_t^2(\xi)} = ???$$

Cylindrical

$$A_t(\xi) = A_t(Y) - \ln(\xi) \delta_t m_0(dr)$$

$$A_t^2(\xi) = A_t^2(Y) - \ln(\xi) \delta_t [A(Y), m_0(dr)]$$

$$- t \ln(\xi)^2 |d_{Y/B} r|^2$$

$$= A_t^2(Y) - t \ln(\xi)^2$$

$$e^{-A_t^2(\xi)} = e^{-A_t^2(Y)} e^{-t \ln(\xi)^2}$$

Proposition (T., 2025)

$$\begin{aligned} [A(Y), m_0(df)] &= m_0 \left(\nabla^{T^* Y} df \right) - 2 \nabla_{(d_{Y/B} f)^\#}^{\mathbb{E}_Y, 0} \\ &= 0 \quad \text{iff the periodic end is cylindrical} \end{aligned}$$