Complex geometry

Alex Taylor
The University of Illinois Urbana-Champaign

Contents

1 Complexification

Let M^n be a real smooth n-dimensional manifold. Any complex-valued function $f \in C^\infty(M;\mathbb{C})$ on M can be uniquely expressed as $f = u + iv$ where $u, v \in C^\infty(M;\mathbb{R})$. Namely, u and v are given by

$$u = \operatorname{Re} f = \frac{1}{2}(f + \overline{f})$$
$$v = \operatorname{Im} f = \frac{1}{2i}(f - \overline{f})$$

Similarly, given a complex-valued vector field Z on M we would like to be able to write $Z = X + iY$ for some real-valued vector fields $X, Y \in \Gamma(M;TM)$. But this expression does not make sense because $Y(p) \in T_pM$ is an element of a real vector space and so the scalar multiplication $iY(p)$ is not well-defined. Thus, in order to make sense of this, we need to extend the real scalar multiplication on each tangent space T_pM to allow for complex scalar multiplication.

There is a natural way of achieving this. Consider the complex vector space $\mathbb{C}$ as an example. A complex number $z = u + iv$ can be identified with an ordered pair $z = (u, v) \in \mathbb{R} \times \mathbb{R}$. Thus $\mathbb{C} \simeq \mathbb{R} \times \mathbb{R}$ as a real 2-dimensional vector space. Under the $\mathbb{R}$-linear isomorphism

$$\mathbb{C} \to \mathbb{R} \times \mathbb{R}$$
$$u + iv \mapsto (u, v)$$

the product $(a + ib) \cdot (u + iv)$ maps to $(au - bv, bu + av)$. Thus, bringing the complex structure back into the picture, as a complex vector space $\mathbb{C}$ is just $\mathbb{R} \times \mathbb{R}$ with complex scalar multiplication given by $(a + ib) \cdot (u, v) = (au - bv, bu + av)$, which turns it into a

1-dimensional complex vector space. Then looking at higher-dimensions, any complex vector $z \in \mathbb{C}^n = \mathbb{C} \times \cdots \times \mathbb{C}$ looks like

$$z = (z_1, z_2, \dots, z_n) = ((u_1, v_1), (u_2, v_2), \dots, (u_n, v_n)) \in \mathbb{R}^n \times \mathbb{R}^n$$

which one can identify with the n-tuple of complex numbers

$$z = (u_1 + iv_1, \dots, u_n + iv_n) = (u_1, \dots, u_n) + i(v_1, \dots, v_n) = u + iv$$

under the aforementioned identification $(u, v) = u + iv$. Once again, the complex scalar multiplication on $\mathbb{C}^n \simeq \mathbb{R}^n \times \mathbb{R}^n$ is given by

$$(a + ib) \cdot (u, v) = (au - bv, bu + av).$$

We can replace $\mathbb{R}^n$ with any abstract real n-dimensional vector space V and follow the same process. Define $V_{\mathbb{C}} = V \times V$ as a real $2n$-dimensional vector space, and then define complex scalar multiplication on $V_{\mathbb{C}}$ by

$$(a + ib) \cdot (u, v) = (au - bv, bu + av).$$

This turns $V_{\mathbb{C}}$ into a complex vector space of complex dimension n, called the ***complexification*** of V. Thus, for example, $\mathbb{C}^n$ is the complexification of $\mathbb{R}^n$.

Fact 1 (Basis for complexification)**.** *Let V be a real vector space with complexification $V_{\mathbb{C}}$. If $(e_1, \dots, e_n)$ is a basis for V over $\mathbb{R}$, then $((e_1, 0), \dots, (e_n, 0))$ is a basis for $V_{\mathbb{C}}$ over $\mathbb{C}$. Under the identification $u + iv = (u, v)$ this basis is just written as $(e_1, \dots, e_n)$ again. In particular this means that $\dim_{\mathbb{R}} V = \dim_{\mathbb{C}} V_{\mathbb{C}}$.*

Proof. Note that for $c = a + ib \in \mathbb{C}$ and $u \in V$ we have $c \cdot (u, 0) = (au, bu) = au + ibu$. Thus if $c^j = a^j + ib^j$ are some complex scalars for which

$$0 = \sum_{j=1}^{n} c^j (e_j, 0) = \sum_{j=1}^{n} (a^j + ib^j)(e_j, 0) = \sum_{j=1}^{n} (a^j e_j, b^j e_j)$$

then this implies that $\sum_j a^j e_j = \sum_j b^j e_j = 0$, which is only possible if $a^j = b^j = 0$ for all j since $\{e_j\}$ is linearly independent over $\mathbb{R}$. Thus $\{(e_j, 0)\}$ is linearly independent over $\mathbb{C}$. Furthermore, given any $w = (u, v) \in V_{\mathbb{C}}$, we can write

$$u = \sum_{j=1}^{n} u^j e_j \quad \text{and} \quad v = \sum_{j=1}^{n} v^j e_j$$

for some scalars $u^j, v^j \in \mathbb{R}$. If we take $w^j = u^j + iv^j$ then

$$\sum_{j=1}^{n} w^j (e_j, 0) = \sum_{j=1}^{n} (u^j e_j, v^j e_j) = (u, v)$$

which shows that $\{(e_j, 0)\}$ is a spanning set for $V_{\mathbb{C}}$. This completes the proof. ∎

In the same way that $\mathbb{R}$ is identified with the real axis $\mathbb{R} \times \{0\} \subseteq \mathbb{C}$, any real vector space V can be identified with the real subspace $V \times \{0\}$ of its complexification $V_{\mathbb{C}}$.

Fact 2. *Let V be a real vector space with complexification $V_{\mathbb{C}}$. Then $V \times \{0\} \subseteq V_{\mathbb{C}}$ is a real subspace which is canonically isomorphic to V (as real vector spaces) under the map*

$$\begin{aligned} &V \to V \times \{0\} \subseteq V_{\mathbb{C}} \\ &u \mapsto (u, 0) \end{aligned}$$

Thus $V \times \{0\}$ is a canonical copy of V inside $V_{\mathbb{C}}$, i.e. V is canonically embedded as a real subspace of $V_{\mathbb{C}}$.

Complexification is a nicely behaved procedure in the sense that it defines a functor from the category of real vector spaces to the category of complex vector spaces. Given real vector spaces V, W and an $\mathbb{R}$-linear map $L : V \to W$, we can extend canonically to a $\mathbb{C}$-linear map $L_{\mathbb{C}} : V_{\mathbb{C}} \to W_{\mathbb{C}}$ by defining

$$L_{\mathbb{C}}(u, v) = (Lu, Lv)$$

i.e. $L(u + iv) = Lu + iLv$ for any $u, v \in V$. This map is indeed complex linear because

$$\begin{aligned} L_{\mathbb{C}}((a + ib)(u, v)) &= L_{\mathbb{C}}(au - bv, bu + av) \\ &= (L(au - bv), L(bu + av)) \\ &= (aL(u) - bL(v), bL(u) + aL(v)) \\ &= (a + ib)(Lu, Lv) \\ &= (a + ib)L_{\mathbb{C}}(u, v). \end{aligned}$$

Evidently $L_{\mathbb{C}}$ is uniquely determined by complex linearity plus the fact that it preserves the two embeddings $u \mapsto (u, 0)$ and $v \mapsto (0, v)$, which is to say that it fits into the two diagrams

$$\begin{array}{ccc} V & \xrightarrow{L} & W \\ {\scriptstyle (u,0)}\big\downarrow & & \big\downarrow{\scriptstyle (w,0)} \\ V_{\mathbb{C}} & \xrightarrow[L_{\mathbb{C}}]{} & W_{\mathbb{C}} \end{array} \qquad \begin{array}{ccc} V & \xrightarrow{L} & W \\ {\scriptstyle (0,u)}\big\downarrow & & \big\downarrow{\scriptstyle (0,w)} \\ V_{\mathbb{C}} & \xrightarrow[L_{\mathbb{C}}]{} & W_{\mathbb{C}} \end{array}$$

Fact 3. *If U, V, W are real vector spaces and $A : V \to W$ and $B : U \to V$ are real-linear maps, then:*

(i) $(A \circ B)_{\mathbb{C}} = A_{\mathbb{C}} \circ B_{\mathbb{C}}$.

(ii) A is invertible if and only if $A_{\mathbb{C}}$ is invertible and $(A^{-1})_{\mathbb{C}} = (A_{\mathbb{C}})^{-1}$.

Remark. Henceforth we will identify a real vector space V with the subspace $V \times \{0\} \subseteq V_{\mathbb{C}}$, and so for any $u \in V$ we will also write $u \in V_{\mathbb{C}}$ to mean $(u, 0)$. Note that this is consistent with the identification $(u, v) = u + iv$. Moreover, for any linear map $L : V \to W$ we will also denote the complexification $L_{\mathbb{C}}$ by L, so for example we will write $L(u + iv) = Lu + iLv$.

The usual notions of conjugation, real part, and imaginary part can be applied to the complexification of any abstract vector space. Given any real vector space V with complexification $V_{\mathbb{C}}$, the ***conjugation*** operator is given by

$$\begin{aligned} &V_{\mathbb{C}} \to V_{\mathbb{C}} \\ &(u, v) \mapsto \overline{(u, v)} = (u, -v) \end{aligned}$$

i.e. $\overline{u+iv} = u - iv$. Evidently this map is $\mathbb{R}$-linear but not $\mathbb{C}$-linear. In fact it is *conjugate-linear*: for any $\lambda \in \mathbb{C}$ and $w \in V_{\mathbb{C}}$ we have $\overline{\lambda \cdot w} = \overline{\lambda} \cdot \overline{w}$. A vector $w \in V_{\mathbb{C}}$ is called ***real*** if $\overline{w} = w$. The real vectors are precisely those vectors $u \in V \subseteq V_{\mathbb{C}}$. Given any vector $w \in V_{\mathbb{C}}$ there are two associated real vectors,

$$\operatorname{Re} w = \frac{1}{2}(w + \overline{w}) \in V \subseteq V_{\mathbb{C}} \text{ called the } \textbf{\textit{real part}} \text{ of } w\,.$$
$$\operatorname{Im} w = \frac{1}{2i}(w - \overline{w}) \in V \subseteq V_{\mathbb{C}} \text{ called the } \textbf{\textit{imaginary part}} \text{ of } w\,.$$

which satisfy $w = \operatorname{Re} w + i \operatorname{Im} w$.

Since complexification is nicely behaved (i.e. functorial) it can be extended immediately from vector spaces to vector bundles. Thus, given a smooth manifold M, we will be able to complexify each tangent space T_pM and assemble a complexified tangent bundle, thereby allowing us to multiply tangent vectors by complex scalars. We start by stating the facts for vector bundles in general. Given a real vector bundle $\pi : E \to M$, the complexification is the complex vector bundle with total space

$$E_{\mathbb{C}} = \bigsqcup_{p \in M} (E_p)_{\mathbb{C}}$$

and with the obvious projection map

$$\begin{aligned} \pi_{\mathbb{C}} &: E_{\mathbb{C}} \to M \\ (p, (u, v)) &\mapsto p \end{aligned}$$

The local data for the complex vector bundle $\pi_{\mathbb{C}} : E_{\mathbb{C}} \to M$ is constructed from that of $\pi : E \to M$ as follows:

(i) Given an open subset $U \subseteq M$ and a local trivialization $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ for E over U, we get a local trivialization $\Phi_{\mathbb{C}} : \pi_{\mathbb{C}}^{-1}(U) \to U \times \mathbb{C}^k$ for $E_{\mathbb{C}}$ over U given by
$$\Phi_{\mathbb{C}}(x, (u, v)) = (x, (\Phi|_{E_x})_{\mathbb{C}}(u, v))$$
i.e. by complexifying the linear isomorphisms on the fibers of $E \to M$.

(ii) Given two overlapping local trivializations (U, Φ) and (V, Ψ) for E with transition map $\tau : U \cap V \to \mathrm{GL}(k, \mathbb{R})$ satisfying
$$(\Psi \circ \Phi^{-1})(x, v) = (x, \tau(x)v)$$
we get a transition map $\tau_{\mathbb{C}} : U \cap V \to \mathrm{GL}(k, \mathbb{C})$ given by $\tau_{\mathbb{C}}(x) = \tau(x)_{\mathbb{C}}$ satisfying
$$(\Psi_{\mathbb{C}} \circ \Phi_{\mathbb{C}}^{-1})(x, (u, v)) = (x, \tau_{\mathbb{C}}(x)(u, v))$$

Thus $\pi_{\mathbb{C}} : E_{\mathbb{C}} \to M$ has a unique structure as a smooth rank-k complex vector bundle, with smooth local trivializations given by the maps $\Phi_{\mathbb{C}}$ defined above.

Fact 4 (Local frames for complexified vector bundle)**.** *Let $E \to M$ be a smooth real vector bundle with complexification $E_{\mathbb{C}} \to M$.*

(i) If $(\sigma_1, \ldots, \sigma_k)$ is a smooth real local frame for E over $U \subseteq M$, then $((\sigma_1, 0), \ldots, (\sigma_k, 0))$ is a smooth complex local frame for $E_{\mathbb{C}}$ over $U \subseteq M$.

(ii) If $(\sigma_1, \ldots, \sigma_k)$ corresponds to a local trivialization Φ for $E|_U$, then $((\sigma_1, 0), \ldots, (\sigma_k, 0))$ corresponds to the local trivialization $\Phi_{\mathbb{C}}$ for $E_{\mathbb{C}}|_U$.

Proof. The fact that $((\sigma_1, 0), \ldots, (\sigma_k, 0))$ constitutes a complex local frame for $E_{\mathbb{C}}$ over $U \subseteq M$ follows immediately from applying Fact 1 pointwise at each $p \in M$ to get a basis for each fiber. Moreover, the smoothness of the local frame will follow from the smoothness of the local trivialization $\Phi_{\mathbb{C}}$, so we just need to show that $((\sigma_1, 0), \ldots, (\sigma_k, 0))$ corresponds to $\Phi_{\mathbb{C}}$.

Say $(\sigma_1, \ldots, \sigma_k)$ corresponds to a local trivialization Φ for E over U, so that

$$\Phi^{-1}(x, (u^1, \ldots, u^k)) = \sum_j u^j \sigma_j(x)$$

for every $x \in U$. Then $((\sigma_1, 0), \ldots, (\sigma_k, 0))$ corresponds to the complexified local trivialization $\Phi_{\mathbb{C}}$ over U. To see why, consider the inverse map

$$\begin{aligned} &\Phi_{\mathbb{C}}^{-1} : U \times \mathbb{C}^k \to E_{\mathbb{C}}|_U \\ &\Phi_{\mathbb{C}}^{-1}(x, w) = (\Phi|_{E_x}^{-1})_{\mathbb{C}}(w) \end{aligned}$$

where for each $x \in U$ we are restricting to the fiber E_x and applying the complexified linear isomorphism

$$(\Phi|_{E_x}^{-1})_{\mathbb{C}} : \mathbb{C}^k \to (E_x)_{\mathbb{C}}.$$

Writing $w = u + iv = (u^1, v^1, \ldots, u^k, v^k) \in \mathbb{C}^k \simeq \mathbb{R}^k \times \mathbb{R}^k$, we have

$$\begin{aligned} \Phi_{\mathbb{C}}^{-1}(x, w) &= (\Phi|_{E_x}^{-1})_{\mathbb{C}}(u, v) \\ &= (\Phi^{-1}(x, u), \Phi^{-1}(x, v)) \\ &= \sum_j (u^j \sigma_j(x)), v^j \sigma_j(x)) \\ &= \sum_j w^j \cdot (\sigma_j(x), 0) \end{aligned}$$

which shows that the local frame $((\sigma_1, 0), \ldots, (\sigma_k, 0))$ corresponds to the local trivialization $\Phi_{\mathbb{C}}$. ∎

Fact 5. *Let $E \to M$ be a smooth real vector bundle with complexification $E_{\mathbb{C}} \to M$. Then the conjugation operator on fibers extends to a smooth conjugate-linear bundle map $c : E_{\mathbb{C}} \to E_{\mathbb{C}}$.*

Proof. Restricting to a fiber over some $p \in M$ we have a conjugate-linear operator

$$c(p, \xi) = (p, \overline{\xi})$$

and gluing these together yields a well-defined conjugate-linear bundle map. In order to show that c is smooth, it suffices to show that $c \circ s \in \Gamma(E_{\mathbb{C}}|_U)$ is smooth for every smooth $s \in \Gamma(E_{\mathbb{C}}|_U)$. Take a smooth local frame $(\sigma^1, \ldots, \sigma^k)$ for E over U and the induced local frame for $((\sigma^1, 0), \dots, (\sigma^k, 0))$ for $E_{\mathbb{C}}$ over U. Then any such $s \in \Gamma(E_{\mathbb{C}}|_U)$ can be expressed as

$$s = \sum_j s^j(\sigma_j, 0)$$

for some smooth component functions $s^j \in C^\infty(M; \mathbb{C})$. Then

$$(c \circ s)(x) = \overline{s(x)} = \sum_j \overline{s^j(\sigma_j, 0)} = \sum_j \overline{s^j(x)}(\sigma_j(x), 0)$$

where the last equality holds because conjugation is conjugate-linear and the vectors $(\sigma_j(x), 0)$ are real. Now if $s^j = a^j + ib^j$ is smooth then $\overline{s^j(x)} = a^j - ib^j$ is also smooth. Thus $c \circ s$ is smooth since it has smooth coefficients in a local frame, and c is a smooth bundle map. ■

Given any local section $s \in \Gamma(M; E_{\mathbb{C}})$ we can write

$$s(x) = (u(x), v(x)) = u(x) + iv(x) \in (E_x)_{\mathbb{C}}$$

for a unique pair of local sections $u, v \in \Gamma(M; E)$. Namely

$$\begin{aligned} u(x) &= \operatorname{Re} s(x) = \frac{1}{2}(s(x) + \overline{s(x)}) \\ v(x) &= \operatorname{Im} s(x) = \frac{1}{2i}(s(x) - \overline{s(x)}) \end{aligned}$$

which are smooth if and only if s is smooth because conjugation is smooth.

Example 1 (Complexified tangent bundle)**.** Let M^n be a smooth manifold. The tangent bundle of M is the smooth real vector bundle of rank n,

$$TM = \bigsqcup_{p \in M} T_pM \to M$$

The complexification of the tangent bundle is the smooth complex vector bundle of rank n,

$$T_{\mathbb{C}}M = \bigsqcup_{p \in M} (T_pM)_{\mathbb{C}} \to M$$

A complex vector field is a section $Z \in \Gamma(M; T_{\mathbb{C}}M)$, i.e. a map

$$\begin{aligned} Z : &M \to T_{\mathbb{C}}M \\ &p \mapsto Z_p = (X_p, Y_p) \in (T_pM)_{\mathbb{C}} = T_pM \times T_pM \end{aligned}$$

which we also write as $Z_p = X_p + iY_p$. Thus the complex vector field Z can be written as $Z = X + iY$ for some real vector fields $X, Y \in \Gamma(M; TM)$. A complex vector field acts as a derivation on smooth complex-valued functions: given $f = u + iv \in C^\infty(M; \mathbb{C})$ we have

$$Z(f) = (X + iY)(u + iv) = (X(u) - Y(v)) + i(X(v) + Y(u)) \in C^\infty(M; \mathbb{C})$$

Example 2 (Complexified cotangent bundle)**.** Let M^n be a smooth manifold. The cotangent bundle of M is the smooth real vector bundle of rank n,

$$T^*M = \bigsqcup_{p\in M} T_p^*M \to M$$

The complexification of the tangent bundle is the smooth complex vector bundle of rank n,

$$T_{\mathbb{C}}^*M = \bigsqcup_{p\in M} (T_p^*M)_{\mathbb{C}} \to M$$

A complex 1-form is a section $\omega \in \Gamma(M; T_{\mathbb{C}}^*M)$, i.e. a map

$$\begin{aligned} \omega :& M \to T_{\mathbb{C}}^*M \\ & p \mapsto \omega_p = (\alpha_p, \beta_p) \in (T_p^*M)_{\mathbb{C}} = T_p^*M \times T_p^*M \end{aligned}$$

which we also write as $\omega_p = \alpha_p + i\beta_p$. Thus the complex 1-form ω can be written as $\omega = \alpha + i\beta$ for some real 1-forms $\alpha, \beta \in \Gamma(M; T^*M)$.

Recall that the dual space of a complex vector space H is by definition the complex vector space H^* consisting of complex-linear functionals on H, i.e.

$$H^* = \mathrm{Hom}_{\mathbb{C}}(H, \mathbb{C}) = \{L : H \to \mathbb{C} : L \text{ is } \mathbb{C}\text{-linear}\}$$

Thus for any $p \in M$ we have $((T_pM)_{\mathbb{C}})^* = \mathrm{Hom}_{\mathbb{C}}((T_pM)_{\mathbb{C}}, \mathbb{C})$. A complex 1-form $\omega \in \Gamma(M; T_{\mathbb{C}}^*M)$ is by definition a map $\omega : M \to T_{\mathbb{C}}^*M$ such that $\omega_p \in (T_p^*M)_{\mathbb{C}}$ for every $p \in M$. Note that ω_p can also be regarded as a linear functional on the complexified tangent space,

$$\omega_p : (T_pM)_{\mathbb{C}} \to \mathbb{C}$$

because we have a natural isomorphism $((T_pM)_{\mathbb{C}})^* \simeq (T_p^*M)_{\mathbb{C}}$ of complex vector spaces.

Fact 6 (Complexification of the dual space)**.** *Let V be a real vector space with complexification $V_{\mathbb{C}}$. Then we have a canonical isomorphism of complex vector spaces*

$$\mathrm{Hom}_{\mathbb{C}}(V_{\mathbb{C}}, \mathbb{C}) = (V_{\mathbb{C}})^* \simeq (V^*)_{\mathbb{C}} = (\mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R}))_{\mathbb{C}}$$

i.e. complexification commutes with dualization.

Proof. It's easy to see that the spaces are isomorphic by comparing their dimensions. A natural isomorphism is also straightforward to describe. An element of $(V^*)_{\mathbb{C}}$ is a pair $(\omega, \eta) = \omega + i\eta$ where $\omega, \eta \in V^*$ are real-linear functionals $V \to \mathbb{R}$. The pair (ω, η) can be regarded as a $\mathbb{C}$-linear functional on $V_{\mathbb{C}} = V \times V$ via

$$(\omega, \eta)(u, v) = (\omega(u) - \eta(v), \omega(v) + \eta(u))$$

i.e. by applying the distributive law to $(\omega + i\eta)(u + iv)$. Thus with this identification we have $(\omega, \eta) \in \mathrm{Hom}_{\mathbb{C}}(V_{\mathbb{C}}, \mathbb{C}) = (V_{\mathbb{C}})^*$. It's straightforward to check that this is a bijective correspondence. ∎

As a consequence, a complex 1-form can be equivalently regarded as a $C^\infty(M;\mathbb{C})$-linear map

$$\begin{aligned}\omega &: \Gamma(M;T_{\mathbb{C}}M) \to C^\infty(M;\mathbb{C})\\ \omega(Z)(p) &= \omega_p(Z_p)\end{aligned}$$

The function $\omega(Z)$ can also be expressed in terms of real and imaginary parts,

$$\omega(Z) = (\alpha + i\beta)(X + iY) = (\alpha(X) - \beta(Y)) + i(\alpha(Y) + \beta(X)).$$

One common way a complex 1-form can appear is as the differential of a smooth complex-valued function. Say we have a smooth function $f : M \to \mathbb{C}$. Then its differential at $p \in M$ is an $\mathbb{R}$-linear map

$$df_p : T_pM \to \mathbb{C}$$

so that $df_p \in \mathrm{Hom}_{\mathbb{R}}(T_pM, \mathbb{C})$. Note that the latter space of functionals can be regarded as a complex vector space by defining complex scalar multiplication by

$$(\lambda \cdot \varphi)(v) = \lambda\varphi(v).$$

for any real-linear functional $\varphi : T_pM \to \mathbb{C}$ and $\lambda \in \mathbb{C}$. Now the real-linear functional df_p yields a complex-linear functional using the following isomorphism:

Fact 7. *Let V be a real vector space with complexification $V_{\mathbb{C}}$. Then we have the following canonical isomorphisms of complex vector spaces:*

$$\mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C}) \simeq \mathrm{Hom}_{\mathbb{C}}(V_{\mathbb{C}}, \mathbb{C})$$

Proof. Given a real-linear map $\varphi : V \to \mathbb{C}$ we can "extend" to a map on $V_{\mathbb{C}}$ by defining

$$\widetilde{\varphi}(u + iv) = \varphi(u) + i\varphi(v)$$

Then $\widetilde{\varphi}$ is complex-linear because

$$\begin{aligned}\widetilde{\varphi}((a+ib)(u+iv)) &= \widetilde{\varphi}((au - bv) + i(av + bu))\\ &= \varphi(au - bv) + i\varphi(av + bu)\\ &= a\varphi(u) + ib\varphi(u) - b\varphi(v) + ia\varphi(v)\\ &= (a + ib)(\varphi(u) + i\varphi(v))\\ &= (a + ib)\widetilde{\varphi}(u + iv)\end{aligned}$$

Thus $\widetilde{\varphi} \in \mathrm{Hom}_{\mathbb{C}}(V_{\mathbb{C}}, \mathbb{C})$. It's straightforward to check that this is a bijective correspondence. ∎

Applying the isomorphism of Fact 7 with $V = T_pM$ we deduce $df_p \in ((T_pM)_{\mathbb{C}})^*$ for every $p \in M$ and so $df \in \Gamma(M; T^*_{\mathbb{C}}M)$ is in fact a complex 1-form. Now suppose that f is decomposed into real and imaginary parts $f = u + iv$. It's natural to expect that we should be able to write $df = du + idv$ with real-valued 1-forms $du, dv \in \Gamma(M; T^*M)$. This claim warrants some justification, though, because a priori df_p is a complex-linear functional on $(T_pM)_{\mathbb{C}}$ whereas du_p and dv_p are real-linear functionals on T_pM. The key to sorting this out is the isomorphism in Fact 6.

First of all note that for real vectors $\xi \in T_pM$ we certainly have

$$df_p(\xi) = du_p(\xi) + idv_p(\xi)$$

simply by definition of the differential df. Moreover, by the isomorphism of Fact 6, the pair $(du_p, dv_p) = du_p + idv_p \in (\mathrm{Hom}_{\mathbb{R}}(T_pM, \mathbb{R}))_{\mathbb{C}}$ is naturally identified with the complex-linear functional $(du_p, dv_p) \in ((T_pM)_{\mathbb{C}})^*$ given by

$$(du_p, dv_p)(\xi + i\zeta) = du_p(\xi) - dv_p(\zeta) + i(du_p(\zeta) + dv_p(\xi)).$$

Thus the action of $df_p \in ((T_pM)_{\mathbb{C}})^*$ looks like

$$\begin{aligned} df_p(\xi + i\zeta) &= df_p(\xi) + idf_p(\zeta) \\ &= du_p(\xi) + idv_p(\xi) + i(du_p(\zeta) + idv_p(\zeta)) \\ &= (du_p(\xi) - dv_p(\zeta)) + i(du_p(\zeta) + dv_p(\xi)) \\ &= (du_p + idv_p)(\xi + i\zeta) \end{aligned}$$

We conclude that it makes sense to write $df = du + idv$ where du and dv are real 1-forms.

2 Complex local coordinates

Let us first consider the model case $\mathbb{C}^n = \mathbb{R}^{2n}$. The standard coordinates are $\{x^1, y^1, \ldots, x^n, y^n\}$, which are global smooth coordinates for $\mathbb{R}^{2n}$ as a smooth $2n$-dimensional manifold. The real 1-forms $(dx^j, dy^j)_{1 \le j \le n}$ constitute a smooth global frame for $T^*\mathbb{R}^{2n}$. Thus by Fact 4 the 1-forms $\{(dx^j, 0), (dy^j, 0)\}_{1 \le j \le n}$ constitute a smooth global frame for the complexified cotangent bundle $T^*_{\mathbb{C}}\mathbb{R}^{2n}$. Recall that we write $(dx^j, dy^j) = dx^j + idy^j$, and any complex 1-form can be expressed as a complex linear combination

$$\sum_j a_j dx^j + b_j dy^j$$

for some complex functions $a_j, b_j \in C^\infty(\mathbb{R}^{2n}, \mathbb{C})$. Consider the complex 1-forms

$$\begin{aligned} dz^j &= dx^j + idy^j \\ d\overline{z}^j &= dx^j - idy^j \end{aligned}$$

The collection $\{dz^j, d\overline{z}^j\}_{1 \le j \le n}$ constitutes another smooth global frame for $T^*_{\mathbb{C}}\mathbb{R}^{2n}$ because we can solve for dx^j, dy^j in terms of $dz^j, d\overline{z}^j$, namely

$$\begin{aligned} dx^j &= \frac{1}{2}(dz^j + d\overline{z}^j) \\ dy^j &= \frac{1}{2i}(dz^j - d\overline{z}^j) \end{aligned}$$

Given a smooth function $f : U \to \mathbb{C}$ its differential is the complex 1-form given by

$$df = \sum_{j=1}^{n} \frac{\partial f}{\partial x^j} dx^j + \frac{\partial f}{\partial y^j} dy^j. \tag{1}$$

We can also express df in terms of the complex frame,

$$df = \sum_{j=1}^{n} A_j dz^j + B_j d\overline{z}^j$$

for some coefficient functions $A_j, B_j \in C^\infty(\mathbb{R}^{2n}, \mathbb{C})$. To see what these coefficients are, we can take equation (1) and replace dx^j and dy^j with their expression in terms of complex differentials,

$$\begin{aligned} df &= \sum_{j=1}^{n} \frac{\partial f}{\partial x^j} dx^j + \frac{\partial f}{\partial y^j} dy^j \\ &= \sum_{j=1}^{n} \frac{\partial f}{\partial x^j} \frac{1}{2}(dz^j + d\overline{z}^j) + \frac{\partial f}{\partial y^j} \frac{1}{2i}(dz^j - d\overline{z}^j) \\ &= \sum_{j=1}^{n} \frac{1}{2}\left(\frac{\partial f}{\partial x^j} - i\frac{\partial f}{\partial y^j}\right) dz^j + \frac{1}{2}\left(\frac{\partial f}{\partial x^j} + i\frac{\partial f}{\partial y^j}\right) d\overline{z}^j \end{aligned}$$

Motivated by this calculation, we define the complex coordinate vector fields on $\mathbb{R}^{2n}$ as

$$\begin{aligned} \frac{\partial f}{\partial z^j} &= \frac{1}{2}\left(\frac{\partial}{\partial x^j} - i\frac{\partial}{\partial y^j}\right) \in \Gamma(T_{\mathbb{C}}\mathbb{R}^{2n}) \\ \frac{\partial f}{\partial \overline{z}^j} &= \frac{1}{2}\left(\frac{\partial}{\partial x^j} + i\frac{\partial}{\partial y^j}\right) \in \Gamma(T_{\mathbb{C}}\mathbb{R}^{2n}) \end{aligned}$$

so that df is then given by

$$df = \sum_{j=1}^{n} \frac{\partial f}{\partial z^j} dz^j + \frac{\partial f}{\partial \overline{z}^j} d\overline{z}^j.$$

When thinking of $\mathbb{R}^{2n}$ as a complex manifold of dimension n, the frame $\{dz^j, d\overline{z}^j\}$ is better because it can be shown that a differential form is *holomorphic* if it has a particular structure with respect to this frame. Thus the complex local coordinates reflect features of the holomorphic structure of a complex manifold.

We recall the definition of local coordinate vector fields on a smooth manifold. Let M^n be a smooth manifold with $\phi : U \subseteq M \to \mathbb{R}^n$ any smooth local chart on M, and let $(x^1, \ldots, x^n)$ denote the associated local coordinates. The chart determines coordinate vector fields $\partial/\partial x^j \in \Gamma(TM|_U)$, $1 \leq j \leq n$, given by

$$\frac{\partial f}{\partial x^j} = \frac{\partial}{\partial x^j}\left(f \circ \phi^{-1}\right)$$

for any smooth function $f \in C^\infty(U; \mathbb{R})$. Then $(\partial/\partial x^1, \ldots, \partial/\partial x^n)$ is a smooth local frame for TM over U.

Now suppose M^{2n} is an even-dimensional smooth manifold, so M is locally homeomorphic to $\mathbb{R}^{2n}$. As before we denote the local coordinates in some chart by $\{x^j, y^j\}_{1 \leq j \leq n}$ and then get local frames

$$\left\{\frac{\partial}{\partial x^j}, \frac{\partial}{\partial y^j}\right\}_{1 \leq j \leq n} \quad \text{for } TM \text{ and } \{dx^j, dy^j\}_{1 \leq j \leq n} \text{ for } T^*M.$$

Since each (co)tangent space is a copy of $\mathbb{R}^{2n}$, we can follow the above procedure to get a local frame for the complexified cotangent bundle $T^*_{\mathbb{C}}M$,

$$dz^j = dx^j + i dy^j \in \Gamma(T^*_{\mathbb{C}}M)$$
$$d\overline{z}^j = dx^j - i dy^j \in \Gamma(T^*_{\mathbb{C}}M)$$

and a local frame for the complexified tangent bundle $T_{\mathbb{C}}M$,

$$\frac{\partial}{\partial z^j} = \frac{1}{2}\left(\frac{\partial}{\partial x^j} - i\frac{\partial}{\partial y^j}\right) \in \Gamma(T_{\mathbb{C}}M)$$
$$\frac{\partial}{\partial \overline{z}^j} = \frac{1}{2}\left(\frac{\partial}{\partial x^j} + i\frac{\partial}{\partial y^j}\right) \in \Gamma(T_{\mathbb{C}}M)$$

These are the ***complex coordinate frames***.

Recall that a smooth function $f : U \subseteq \mathbb{C} \to \mathbb{C}$ is ***holomorphic*** if it satisfies the Cauchy-Riemann equations. If $f(x, y) = u(x, y) + iv(x, y)$ then the Cauchy-Riemann equations are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

We can express these equations in terms of the complex partial derivative $\partial/\partial\overline{z} = \partial/\partial x + i\partial/\partial y$. We calculate

$$\begin{aligned}
\frac{\partial f}{\partial \overline{z}} &= \frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y} \\
&= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) + i\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right) \\
&= \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + i\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)
\end{aligned}$$

and thus we see that

$$\text{CR equations for } f \quad \leftrightarrow \quad \frac{\partial f}{\partial \overline{z}} \equiv 0$$

More generally for a smooth function $f : U \subseteq \mathbb{C}^n \to \mathbb{C}$, we say that f is holomorphic if it satisfies the Cauchy-Riemann equations in each variables z^j. Thus we have the following important characterization.

Fact 8 (Holomorphic functions)**.** *A smooth function $f : U \subseteq \mathbb{C}^n \to \mathbb{C}$ is holomorphic if and only if*

$$\frac{\partial f}{\partial \overline{z}^j} \equiv 0 \text{ on } U$$

for every $j = 1, \ldots, n$.

3 Almost complex structures

Let V be a real vector space. A ***complex structure*** on V is a real-linear endomorphism $J : V \to V$ such that $J^2 = -I$. Evidently J is supposed to be something like a "multiplication by i" operator.

Fact 9. *Let V be a real vector space and let $J : V \to V$ be a complex structure on V. Then:*

(i) V can be turned into a complex vector space by defining $i \cdot v = Jv$ for any $v \in V$ and extending linearly, i.e.

$$(a + ib) \cdot v = av + bJv$$

for any $v \in V$. We denote this complex vector space by V_J.

(ii) V has even real dimension $\dim_{\mathbb{R}} V = 2 \dim_{\mathbb{C}} V_J$.

Proof. (i) Note that the definition $i \cdot v = Jv$ makes sense because $J^2 = -I$ and thus $-v = i^2 \cdot v = J^2 v = -v$ is consistent. It is straightforward to check that complex multiplication so defined is associative and distributes over addition.

(ii) Say V_J has complex dimension $n \geq 1$ let $\{v_1, \ldots, v_n\}$ be any basis for V_J over $\mathbb{C}$. Then any $v \in V_J$ can be expressed as

$$v = \sum_{j=1}^{n} c_j v_j$$

for some complex coefficients $c_j \in \mathbb{C}$, say $c_j = a_j + ib_j$. Then

$$v = \sum_{j=1}^{n} c_j v_j = \sum_{j=1}^{n} (a_j + ib_j) v_j = \sum_{j=1}^{n} a_j v_j + b_j J v_j$$

and therefore $\{v_1, \ldots, v_n, Jv_1, \ldots, Jv_n\}$ is a spanning set for V over $\mathbb{R}$. This set is also linearly independent over $\mathbb{R}$ because $\{v_1, \ldots, v_n\}$ is linearly independent over $\mathbb{C}$. Thus $\{v_1, \ldots, v_n, Jv_1, \ldots, Jv_n\}$ is a basis for V over $\mathbb{R}$.

■

So far we have discussed two ways of turning V into a complex vector space: V_J (underlying real space V) and $V_{\mathbb{C}}$ (underlying real space $V \times V$). It turns out that there is an important relationship between V_J and $V_{\mathbb{C}}$. Complexify J to get a complex-linear map $J_{\mathbb{C}} : V_{\mathbb{C}} \to V_{\mathbb{C}}$ satisfying $J_{\mathbb{C}}^2 = -I$ where I now denotes the identity map on $V_{\mathbb{C}}$. Then the eigenvalues of $J_{\mathbb{C}}$ satisfy $\lambda^2 = -1$ so they are $\lambda = \pm i$.

Fact 10. *Let J be a complex structure on a real vector space V and let $V_{\mathbb{C}}$ denote the complexification of V. Then V_J and V_{-J} are complex subspaces of $V_{\mathbb{C}}$, namely*

$$V_J \simeq V' = +i \text{ eigenspace of } J_{\mathbb{C}}$$
$$V_{-J} \simeq V'' = -i \text{ eigenspace of } J_{\mathbb{C}}$$

and moreover, we have a complete eigenspace decomposition

$$V_{\mathbb{C}} = V' \oplus V''.$$

whereby any $w \in V_{\mathbb{C}}$ decomposes into $w = w' + w''$, with

$$w' = \frac{1}{2}(w - iJ_{\mathbb{C}} w), \qquad w'' = \frac{1}{2}(w + iJ_{\mathbb{C}} w).$$

Proof. First we show that $V_J \simeq V'$. Consider the complex-linear map

$$\varphi : V_J \to V_{\mathbb{C}}$$
$$\varphi(v) = v - iJv = (v, -Jv)$$

Note that

$$J_{\mathbb{C}}\varphi(v) = J(v - iJv) = Jv - iJ^2 v = Jv + iv = i\varphi(v)$$

so $\varphi(v) \in V' \subseteq V_{\mathbb{C}}$ for every $v \in V_J$. Moreover, φ is injective, because if $v \in V_J$ such that $\varphi(v) = 0$ then $v = iJv$ i.e. $Jv = -iv$, but $v \in V_J$ implies that $Jv = iv$ as well. So $Jv = 0$ and since J is invertible we have $v = 0$. Thus φ is a complex-linear isomorphism. A similar argument using the map $v \mapsto v + iJv$ shows that $V_{-J} \simeq V''$.

Regarding the eigenspace decomposition $V_{\mathbb{C}} = V' \oplus V''$: an argument similar to the above shows that $w' \in V'$ and $w'' \in V''$, and it's clear that they satisfy $w = w' + w''$. On the other hand, a nonzero vector cannot be an eigenvector for two distinct eigenvalues so $V' \cap V'' = 0$ and this completes the proof. ∎

Fact 11. *The conjugation operator on $V_{\mathbb{C}}$ interchanges V' and V'', thus it defines a real-linear isomorphism of the underlying real vector spaces, and*

$$\dim_{\mathbb{C}} V' = \dim_{\mathbb{C}} V'' = \frac{1}{2}\dim_{\mathbb{C}} V_{\mathbb{C}}.$$

Let's illustrate Fact 10 in the case that $V = \mathbb{R}^{2n}$. The standard basis for $\mathbb{R}^{2n}$ is the set of vectors $\{X_1, \ldots, X_n, Y_1, \ldots, Y_n\}$ where

$$X_j = (0, \ldots, 1, \ldots, 0) \quad \text{(1 in the } j\text{th entry)}$$
$$Y_j = (0, \ldots, 1, \ldots, 0) \quad \text{(1 in the } (n+j)\text{th entry)}$$

and with respect to this basis the standard complex structure is the $2n \times 2n$ matrix

$$J_{\mathbb{R}^{2n}} = \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix}$$

where I_n is the $n \times n$ identity matrix. Evidently then

$$JX_j = Y_j, \;\; JY_j = -X_j.$$

Consider the complexification $J_{\mathbb{C}} : (\mathbb{R}^{2n})_{\mathbb{C}} \to (\mathbb{R}^{2n})_{\mathbb{C}}$ (a complex-linear map between complex vector spaces of dimension $2n$). From Fact 10, the eigenspaces of $J_{\mathbb{C}}$ are given by

$$(\mathbb{R}^{2n})' = \operatorname{span}\{X_j - iJX_j : 1 \le j \le n\} = \operatorname{span}\{X_j - iY_j : 1 \le j \le n\}$$
$$(\mathbb{R}^{2n})'' = \operatorname{span}\{X_j + iJX_j : 1 \le j \le n\} = \operatorname{span}\{X_j + iY_j : 1 \le j \le n\}.$$

Now suppose M is a smooth manifold of real dimension $2n$. For any point $p \in M$ we can take a smooth chart (U, ϕ) centered at $x \in M$ which affords linear isomorphisms

$$d\phi_p : T_pM \xrightarrow{\simeq} \mathbb{R}^{2n}$$
$$(d\phi_p)_{\mathbb{C}} : (T_pM)_{\mathbb{C}} \xrightarrow{\simeq} (\mathbb{R}^{2n})_{\mathbb{C}}$$

by which we identify

$$\left.\frac{\partial}{\partial x^j}\right|_p \leftrightarrow X_j,\ \left.\frac{\partial}{\partial y^j}\right|_p \leftrightarrow Y_j,\ \left.\frac{\partial}{\partial z^j}\right|_p \leftrightarrow Z_j = \frac{1}{2}(X_j - iY_j)$$

Under this identification, the standard complex structure on $\mathbb{R}^{2n}$ gives us a complex structure on the tangent space T_pM. Thus we have a decomposition $(T_pM)_{\mathbb{C}} = T_p'M \oplus T_p''M$ where $T_p'M$ and $T_p''M$ are the subspaces spanned by

$$T_p'M = \operatorname{span}\left\{\left.\frac{\partial}{\partial z^j}\right|_p\right\}_{1\le j\le n},\quad T_p''M = \operatorname{span}\left\{\left.\frac{\partial}{\partial \overline{z}^j}\right|_p\right\}_{1\le j\le n}.$$

We would like to glue these fiberwise complex structures together to construct a complex structure on the tangent bundle $TM \to M$, i.e. a smooth real-linear bundle endomorphism $J : TM \to TM$ satisfying $J^2 = -I$. Taking a smooth chart $\phi : U \subseteq M \to \mathbb{R}^{2n}$, we should try to define

$$\begin{aligned} &J : TM|_U \to TM|_U \\ &J = D\phi^{-1} \circ J_{\mathbb{R}^{2n}} \circ D\phi \end{aligned}$$

In order for this to yield a well-defined global endomorphism of TM, the endomorphisms need to agree on the overlap of two different smooth charts (U, ϕ) and (V, ψ) on M. Thus we need

$$D\phi^{-1} \circ J_{\mathbb{R}^{2n}} \circ D\phi = D\psi^{-1} \circ J_{\mathbb{R}^{2n}} \circ D\psi$$

on the overlap $U \cap V$. We can rewrite both sides of this equation as follows:

$$\begin{aligned} D\phi^{-1} \circ J_{\mathbb{R}^{2n}} \circ D\phi &= (D\psi^{-1} \circ D\psi) \circ D\phi^{-1} \circ J_{\mathbb{R}^{2n}} \circ D\phi \\ &= D\psi^{-1} \circ (D\psi \circ D\phi^{-1}) \circ J_{\mathbb{R}^{2n}} \circ D\phi \end{aligned}$$

and on the other side

$$\begin{aligned} D\psi^{-1} \circ J_{\mathbb{R}^{2n}} \circ D\psi &= D\psi^{-1} \circ J_{\mathbb{R}^{2n}} \circ D\psi \circ (D\phi^{-1} \circ D\phi) \\ &= D\psi^{-1} \circ J_{\mathbb{R}^{2n}} \circ (D\psi \circ D\phi^{-1}) \circ D\phi \end{aligned}$$

Thus we see that, in order for these two expressions to coincide, we need to be able to commute the complex structure $J_{\mathbb{R}^{2n}}$ with the differential $D(\psi \circ \phi^{-1})$ of the transition map, i.e. we need to have

$$D(\psi \circ \phi^{-1}) \circ J_{\mathbb{R}^{2n}} = J_{\mathbb{R}^{2n}} \circ D(\psi \circ \phi^{-1}).$$

If this holds for every pair of overlapping charts on M, then the complex structures glue together to produce a well-defined global complex structure $J_M : TM \to TM$. We will call this the ***canonical complex structure*** on TM because it arises in a natural way from the smooth structure of M and the standard complex structure of $\mathbb{R}^{2n}$. We summarize this observation:

Fact 12. *Let M be a smooth manifold of real dimension $2n$. Then TM admits a canonical complex structure $J_M : TM \to TM$ if and only if*

$$D(\psi \circ \phi^{-1}) \circ J_{\mathbb{R}^{2n}} = J_{\mathbb{R}^{2n}} \circ D(\psi \circ \phi^{-1})$$

for every pair of overlapping charts (U, ϕ) and (V, ψ) on M.

Remark. A few important remarks regarding terminology.

1. The term "canonical" complex structure in Fact 12 is an important detail. In general there may be many different complex structures on TM, and it is possible for the commutator condition to fail while TM admits a complex structure which is *different* from the canonical one. Given a smooth manifold M and a complex structure J on TM can we determine if it's canonical? Are there obstructions against constructing a canonical complex structure on TM? This is a nontrivial question in general.

2. A complex structure on TM is also called an "almost complex structure" on M. Thus an "almost complex manifold" is a smooth manifold M together with a complex structure $J : TM \to TM$ on TM.

3. A complex manifold is a smooth manifold M equipped with an additional holomorphic structure so that the transition maps are holomorphic. This is stronger than being an almost complex manifold. We will show below that a complex manifold admits a canonical almost complex structure.

In order to demonstrate the significance of Fact 12 we will need to develop some basic ideas about the complex differential of a smooth function. Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be a smooth map and let $dF : T\mathbb{R}^n \to T\mathbb{R}^m$ denote its differential. Its complexification is the complex-linear bundle homomorphism

$$DF = (dF)_{\mathbb{C}} : T_{\mathbb{C}}\mathbb{R}^n \to T_{\mathbb{C}}\mathbb{R}^m$$

and we call DF the ***complex differential*** of F.

Fact 13 (Complex differential in coordinates)**.** *Let $F : \mathbb{R}^{2n} \to \mathbb{R}^{2m}$ be a smooth map. Let $\{\partial/\partial z^j, \partial/\partial \overline{z}^j\}_{1\leq j\leq n}$ denote the complex coordinate frame for $T_{\mathbb{C}}\mathbb{R}^{2n}$ and $\{\partial/\partial w^k, \partial/\partial \overline{w}^k\}_{1\leq k\leq m}$ the complex coordinate frame for $T_{\mathbb{C}}\mathbb{R}^{2m}$. Then the complex differential is given by:*

$$DF\left(\frac{\partial}{\partial z^j}\right) = \sum_k \frac{\partial F^k}{\partial z^j}\frac{\partial}{\partial w^k} + \frac{\partial \overline{F}^k}{\partial z^j}\frac{\partial}{\partial \overline{w}^k}$$
$$DF\left(\frac{\partial}{\partial \overline{z}^j}\right) = \sum_k \frac{\partial F^k}{\partial \overline{z}^j}\frac{\partial}{\partial w^k} + \frac{\partial \overline{F}^k}{\partial \overline{z}^j}\frac{\partial}{\partial \overline{w}^k}$$

Proof. Say $\{x^j, y^j\}$ are standard real coordinates on $\mathbb{R}^{2n}$ and $\{u^k, v^k\}$ are standard real coordinates on $\mathbb{R}^{2m}$. The corresponding complex coordinates are $z^j = x^j + iy^j$ and $w^k = u^k + iv^k$ and the complex coordinate frames are

$$\frac{\partial}{\partial z^j} = \frac{1}{2}\left(\frac{\partial}{\partial x^j} - i\frac{\partial}{\partial y^j}\right), \quad \frac{\partial}{\partial \overline{z}^j} = \frac{1}{2}\left(\frac{\partial}{\partial x^j} + i\frac{\partial}{\partial y^j}\right)$$
$$\frac{\partial}{\partial w^k} = \frac{1}{2}\left(\frac{\partial}{\partial u^k} - i\frac{\partial}{\partial v^k}\right), \quad \frac{\partial}{\partial \overline{w}^k} = \frac{1}{2}\left(\frac{\partial}{\partial u^k} + i\frac{\partial}{\partial v^k}\right)$$

Thus we have

$$\begin{aligned}\frac{\partial}{\partial u^k} &= \frac{\partial}{\partial w^k} + \frac{\partial}{\partial \overline{w}^k}\\ \frac{\partial}{\partial v^k} &= i\left(\frac{\partial}{\partial w^k} - \frac{\partial}{\partial \overline{w}^k}\right)\end{aligned}$$

Writing $F = U + iV$ in complex coordinates, by definition the differential of F is given by

$$\begin{aligned}dF\left(\frac{\partial}{\partial x^j}\right) &= \sum_k \frac{\partial U^k}{\partial x^j}\frac{\partial}{\partial u^k} + \frac{\partial V^k}{\partial x^j}\frac{\partial}{\partial v^k}\\ dF\left(\frac{\partial}{\partial y^j}\right) &= \sum_k \frac{\partial U^k}{\partial y^j}\frac{\partial}{\partial u^k} + \frac{\partial V^k}{\partial y^j}\frac{\partial}{\partial v^k}\end{aligned}$$

Thus by complex-linearity the complex differential is given by

$$DF\left(\frac{\partial}{\partial z^j}\right) = \sum_k \frac{\partial U^k}{\partial z^j}\frac{\partial}{\partial u^k} + \frac{\partial V^k}{\partial z^j}\frac{\partial}{\partial v^k}.$$

Now we convert this into the complex coordinate frame on $T_{\mathbb{C}}\mathbb{R}^{2m}$

$$\begin{aligned}DF\left(\frac{\partial}{\partial z^j}\right) &= \sum_k \frac{\partial U^k}{\partial z^j}\left(\frac{\partial}{\partial w^k} - \frac{\partial}{\partial \overline{w}^k}\right) + \frac{\partial V^k}{\partial z^k}\cdot i\left(\frac{\partial}{\partial w^k} - \frac{\partial}{\partial \overline{w}^k}\right)\\ &= \sum_k \left(\frac{\partial U^k}{\partial z^j} + i\frac{\partial V^k}{\partial z^j}\right)\frac{\partial}{\partial w^k} + \left(\frac{\partial U^k}{\partial z^j} - i\frac{\partial V^k}{\partial z^j}\right)\frac{\partial}{\partial \overline{w}^k}\\ &= \sum_k \frac{\partial F^k}{\partial z^j}\frac{\partial}{\partial w^k} + \frac{\partial \overline{F}^k}{\partial z^j}\frac{\partial}{\partial \overline{w}^k}\end{aligned}$$

which is the formula we wanted to establish. The calculation for $DF(\partial/\partial\overline{z}^j)$ is similar. ■

Fact 14. *Let $U \subseteq \mathbb{R}^{2n}$ be an open subset and let $F : U \subseteq \mathbb{R}^{2n} \to \mathbb{R}^{2m}$ be a smooth map. Then F is holomorphic as a map $F : U \subseteq \mathbb{C}^n \to \mathbb{C}^m$ if and only if*

$$DF(p) \circ J_{\mathbb{C}^n} = J_{\mathbb{C}^m} \circ DF(p) \tag{2}$$

holds for every $p \in U$.

Proof. First suppose equation (2) holds on U. Then using the complex coordinate frame expression for DF we calculate

$$\begin{aligned}
0 &= DF\left(J_{\mathbb{C}^n}\frac{\partial}{\partial \overline{z}^j}\right) - J_{\mathbb{C}^m}\left(DF\left(\frac{\partial}{\partial \overline{z}^j}\right)\right) \\
&= DF\left(-i\frac{\partial}{\partial \overline{z}^j}\right) - J_{\mathbb{C}^m}\left(\sum_k \frac{\partial F^k}{\partial \overline{z}^j}\frac{\partial}{\partial w^k} + \frac{\partial \overline{F}^k}{\partial \overline{z}^j}\frac{\partial}{\partial \overline{w}^k}\right) \\
&= -iDF\left(\frac{\partial}{\partial \overline{z}^j}\right) - \sum_k \frac{\partial F^k}{\partial \overline{z}^j} J_{\mathbb{C}^m}\left(\frac{\partial}{\partial w^k}\right) + \frac{\partial \overline{F}^k}{\partial \overline{z}^j} J_{\mathbb{C}^m}\left(\frac{\partial}{\partial \overline{w}^k}\right) \\
&= -iDF\left(\frac{\partial}{\partial \overline{z}^j}\right) - \sum_k \frac{\partial F^k}{\partial \overline{z}^j}\cdot i\frac{\partial}{\partial w^k} - \frac{\partial \overline{F}^k}{\partial \overline{z}^j}\cdot i\frac{\partial}{\partial \overline{w}^k} \\
&= -i\sum_k \frac{\partial F^k}{\partial \overline{z}^j}\frac{\partial}{\partial w^k} + \frac{\partial \overline{F}^k}{\partial \overline{z}^j}\frac{\partial}{\partial \overline{w}^k} - \sum_k \frac{\partial F^k}{\partial \overline{z}^j}\cdot i\frac{\partial}{\partial w^k} - \frac{\partial \overline{F}^k}{\partial \overline{z}^j}\cdot i\frac{\partial}{\partial \overline{w}^k} \\
&= -2i\sum_k \frac{\partial F^k}{\partial \overline{z}^j}\frac{\partial}{\partial w^k}
\end{aligned}$$

hence $\partial F^k/\partial \overline{z}^j \equiv 0$ on U for every j, k, which implies that F is holomorphic.

Conversely, suppose that F is holomorphic. Then $\partial F^k/\partial \overline{z}^j \equiv 0$ for every j, k and so the above calculation shows that equation (2) holds when applied to $\partial/\partial \overline{z}^j$. Then by conjugating everything we find that the equation also holds when applied to $\partial/\partial z^j$. Thus (2) is satisfied. ■

Applying this fact the transition map $F = \psi \circ \phi^{-1}$ determined by two overlapping charts on M, together with Fact 12, we conclude that TM admits a canonical complex structure $J_M : TM \to TM$ if and only if the transition maps are holomorphic. Thus, if M is a complex manifold (i.e. a smooth manifold equipped with an additional holomorphic structure so that the transition maps are holomorphic) then TM admits a canonical complex structure $J_M : TM \to TM$.

Corollary 1. *Let M be a complex manifold. Then TM admits a canonical complex structure $J_M : TM \to TM$.*

Let M be a smooth manifold of real dimension $2n$ with almost complex structure $J : TM \to TM$. Then we can replicate the eigenspace decomposition from Fact 10 at the level of bundles. Complexifying yields a complex bundle endomorphism $J_{\mathbb{C}} : T_{\mathbb{C}}M \to T_{\mathbb{C}}M$ with $\pm i$-eigenspaces T'_pM and T''_pM in each fiber, and so we can define smooth complex subbundles

$$T'M = \bigsqcup_{p\in M} T'_pM \subseteq T_{\mathbb{C}}M \to M$$

$$T''M = \bigsqcup_{p\in M} T''_pM \subseteq T_{\mathbb{C}}M \to M$$

$T'M$ is called the ***holomorphic tangent bundle*** and $T''M$ the ***antiholomorphic tangent bundle***.

Fact 15. *Let M^{2n} be a smooth manifold with almost complex structure $J : TM \to TM$. Let $T'M$ and $T''M$ denote the holomorphic and antiholomorphic tangent bundles associated with J. Then we have a decomposition*

$$T_{\mathbb{C}}M = T'M \oplus T''M.$$

Proof. The proof is similar to the proof of Fact 10. ■

If J is merely an arbitrary almost complex structure on M then this is about all we can say; however, if J is the canonical almost complex structure induced by a holomorphic structure on M, then we can also say that the complex local coordinate frame for $T_{\mathbb{C}}M$ splits into two local frames for $T'M$ and $T''M$.

Fact 16. *Let M be a smooth manifold with canonical almost complex structure $J_M : TM \to TM$. Let $T'M$ and $T''M$ denote the holomorphic and antiholomorphic tangent bundles associated with J_M. If $\{z^j\}$ are local complex coordinates on M then we obtain local frames*

$$\left\{\frac{\partial}{\partial z^j}\right\}_{1\leq j\leq n} \text{ for } T'M, \ \left\{\frac{\partial}{\partial \overline{z}^j}\right\}_{1\leq j\leq n} \text{ for } T''M.$$

Proof. Fix a smooth local chart (U, ϕ) on M and local complex coordinates $\{z^j\}$ on $U \subseteq M$. Then the canonical almost complex structure J_M is built up from the standard complex structure on $\mathbb{R}^{2n}$, via $J_M = D\phi^{-1} \circ J_{\mathbb{R}^{2n}} \circ D\phi$ on $TM|_U$. Using this we can calculate

$$\begin{aligned} J_M \frac{\partial}{\partial z^j} &= D\phi^{-1} J_{\mathbb{R}^{2n}} D\phi \frac{\partial}{\partial z^j} \\ &= D\phi^{-1}\left(J_{\mathbb{R}^{2n}} Z_j\right) \\ &= iD\phi^{-1}(Z_j) \\ &= i\frac{\partial}{\partial z^j} \end{aligned}$$

which shows that $\partial/\partial z^j \in \Gamma(T'M|_U)$. Since the collection gives a basis for each fiber, it follows that this is a local frame for $T'M$ (see the discussion following Fact 11 at the level of fibers). The calculation for $\{\partial/\partial\overline{z}_j\}$ is similar. Note that, when J is merely an arbitrary almost complex structure on M, we do not know a priori how J interacts with the local frame $\{\partial/\partial z^j, \partial/\partial\overline{z}^j\}$ for $T_{\mathbb{C}}M$. ■

Recall from before we had $V_J \simeq V'$. We can replicate this at the level of bundles.

Fact 17. *Let M^{2n} be a smooth manifold with almost complex structure $J : TM \to TM$. Then we have a smooth bundle isomorphism*

$$\begin{aligned} &\varphi : T_J M \to T'M \\ &\varphi(v) = v - iJv \end{aligned}$$

In summary, for a smooth manifold M^{2n} with almost complex structure $J : TM \to TM$, we have the following smooth complex vector bundles over M:

1. $T_{\mathbb{C}}M$: the complexified tangent bundle. Complex rank $2n$. Fibers are complexified tangent spaces $(T_pM)_{\mathbb{C}} = T_pM \times T_pM$.

2. $T'M$: the holomorphic tangent bundle. Complex rank n subbundle of $T_{\mathbb{C}}M$. Fibers are $+i$-eigenspaces of $J_{\mathbb{C}}$.

3. $T''M$: the antiholomorphic tangent bundle. Complex rank n subbundle of $T_{\mathbb{C}}M$. Fibers are $-i$-eigenspaces of $J_{\mathbb{C}}$.

4. T_JM: same underlying smooth manifold as TM, but each fiber is T_pM regarded as a complex vector space with complex multiplication given by the action of J. Complex rank n bundle isomorphic to $T'M$.

Note that TM and $T_{\mathbb{C}}M$ make sense with the smooth structure of M only, but the other three involve the almost complex structure J in their definition.

Remark. Suppose M is a smooth manifold with an almost complex structure $J : TM \to TM$? Natural to ask: can we construct a holomorphic structure on M which induces J as the associated (canonical) almost complex structure? The Newlander-Nirenberg theorem says that this is the case if and only if the almost complex structure J is integrable.