Shape Parameters of Ideal Tetrahedra

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The goal of this report is to define the notion of ideal tetrahedron and the shape parameters associated with its edges in hyperbolic space, prove some basic facts about shape parameters, and then conclude with a brief description of how these concepts are typically used in hyperbolic geometry (by Thurston, for example).

Let $\mathbb{H}^3$ denote the three-dimensional hyperbolic space. An **ideal tetrahedron** T in $\mathbb{H}^3$ is the convex hull of four points (the vertices of T), all of which lie on the sphere at infinity $S^2_\infty \subset \mathbb{H}^3$. In other words, an ideal tetrahedron is a tetrahedron whose vertices are ideal points. We picture this object as a (hyperbolically) distorted Euclidean tetrahedron whose vertices have been stretched to infinity (Figure 1).

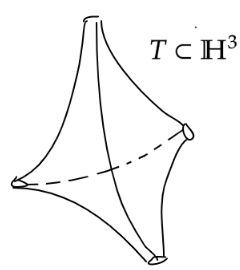

Figure 1: An ideal tetrahedron.

Now consider a horospherical cross-section of one of the cusps of T (Figure 2). The cross-section of this cusp produces an honest Euclidean triangle, say with vertices $v_1, v_2, v_3 \in \mathbb{C}$ and therefore we define the **dihedral angles** of T to be the complex numbers $z_1, z_2, z_3 \in \mathbb{C} \setminus \{0, 1, \infty\}$ such that

$$\begin{aligned} \operatorname{Arg}(z_i) &= \text{angle at } v_i, \\ |z_i| &= \text{ratio of the lengths of the sides incident to } v_i. \end{aligned}$$

In other words, z_1 is the complex number which rotates the side $v_3 - v_1$ counterclockwise onto $v_2 - v_1$, and similarly for the other two dihedral angles. Now we define the **shape parameter** of the edge $e \in T$ to be the dihedral angle $z(e)$ of the vertex lying on e in the horospherical cross-section.

A priori, we do not really know yet that the shape parameter of any given edge in T is well-defined – technically speaking have associated two dihedral angles to each edge, one for each "half" of the edge. We will prove shortly that in fact these two dihedral angles coincide, which shows that the shape parameter is in fact well-defined.

Remark. There is another issue which potentially obstructs the well-definedness of the shape parameter. Is there a unique horospherical cross-section of any cusp of T? If not, we would need to show that all of the different horospherical cross-sections produce similar triangles, so that the shape parameters are independent of the choice of cross-section. I do not attempt to resolve this technicality in this report.

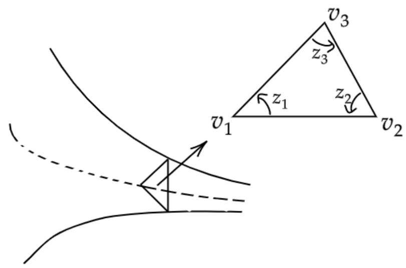

Figure 2: The horospherical cross section with a cusp of T yields a Euclidean triangle, and the dihedral angles z_i can be associated to each vertex of this triangle.

Fact 1. Any one of the three dihedral angles determines the other two.

Proof. This is a straightforward computation which follows directly from the definition of the dihedral angles. The complex number z_1 which rotates $v_3 - v_1$ onto $v_2 - v_1$ is exactly

$$z_1 = \frac{v_2 - v_1}{v_3 - v_1}$$

and similarly

$$\begin{aligned} z_2 &= \frac{v_3 - v_2}{v_1 - v_2} \\ z_3 &= \frac{v_1 - v_3}{v_2 - v_3} \end{aligned}$$

so we calculate

$$\begin{aligned} z_2 &= \frac{v_3 - v_2}{v_1 - v_2} \\ &= \frac{v_3 - v_1 + v_1 - v_2}{v_1 - v_2} \\ &= \frac{v_3 - v_1}{v_1 - v_2} + 1 \\ &= 1 - \frac{1}{z_1} \end{aligned}$$

and also

$$\begin{aligned} z_3 &= \frac{v_1 - v_3}{v_2 - v_3} \\ &= \frac{v_1 - v_2 + v_2 - v_3}{v_2 - v_3} \\ &= \frac{v_1 - v_2}{v_2 - v_3} + 1 \\ &= \frac{1}{z_2} + 1 \\ &= \frac{1}{1 - z_1} \end{aligned}$$

so we can solve for z_2 and z_3 directly in terms of z_1. ∎

Fact 2. Opposite edges have the same shape parameter.

Proof. We follow Thurston's argument by symmetry in [1]. For any pair of opposite edges in T, the shortest geodesic between them must meet both of the edges at right angles – thus the perpendicular bisector joining the opposite edges is an axis of symmetry of T (Figure 3). Since there are three pairs of opposite edges we obtain three axes of symmetry. The group of permutations of the four vertices of T via rotational symmetries is A_4; explicitly, we can describe the elements of A_4 as rotations in the following way:

1. The order 2 elements $\{(12)(34), (14)(23), (13)(24)\}$ can each be associated with a pair of opposite edges (hence an axis of symmetry) – these elements rotate T by π about the axis. If we focus on one of the edges bisected by the axis of symmetry, we see that the effect of the rotation is to simply interchange the vertices of this edge.

2. The order 3 elements $\{(123), (132), (124), (142), (234), (243), (134), (314)\}$ fix one vertex of T and then spin T by $2\pi/3$ around the line joining the vertex to its opposite face.

Let σ be an order 2 element of A_4, i.e. σ rotates by π about an axis of symmetry. Then σ preserves the pair of edges bisected by this axis, hence it preserves their shape parameters. As a result, the shape parameters are preserved by the action of the subgroup of order 2 elements of A_4,

$$\mathbb{Z}_2 \times \mathbb{Z}_2 \simeq \{\text{id}, (12)(34), (14)(23), (13)(24)\}.$$

A priori, we could assign a different shape parameter to each half-edge of T. There are 12 half-edges, so we have a correspondence between shape parameters of T and elements of A_4. On the other hand, since $\mathbb{Z}_2 \times \mathbb{Z}_2 \subset A_4$ preserves the shape parameters, we don't lose any information by taking the quotient

$$A_4/(\mathbb{Z}_2 \times \mathbb{Z}_2) \simeq \mathbb{Z}_3 \simeq \{\text{pairs of opposite edges of } T\},$$

thus we obtain a one-to-one correspondence

$$\{\text{shape parameters of } T\} \simeq \{\text{pairs of opposite edges of } T\}.$$

In other words, opposite pairs of edges have the same shape parameters. ∎

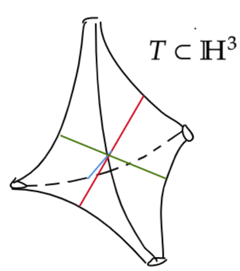

Figure 3: Three pairs of opposite edges yield three perpendicular bisectors, which are axes of symmetry around which the subgroup $Z_2 \times Z_2 \subset A_4$ rotates T.

As a corollary, we can use **Fact 1** and **Fact 2** to deduce the well-definedness of the shape parameter, i.e. there is a unique number associated to each edge. Fix an edge e on the ideal tetrahedron T and consider two triangles which determine a shape parameter z_1 for one half of e and w_1 for the other half. Then z_3 and w_3 are shape parameters for opposite edges, so $z_3 = w_3$ by **Fact 2**. But then

$$z_1 = 1 - z_3^{-1} = 1 - w_3^{-1} = w_1$$

by the calculations from **Fact 1**, so the shape parameters assigned to opposite halves of e coincide. In the words of Jeff Weeks, "a single complex dihedral angle completely parameterizes the shape of an ideal tetrahedron." Since a single complex number $z \in \mathbb{C} \setminus \{0, 1, \infty\}$ can be used to describe the parameters of all of the edges of an ideal tetrahedron T, we can simply say that z is the shape parameter of T. We have yet to determine how the shape parameter can be calculated from the vertices of T. In fact, it is straightforward to prove that for the tetrahedron with vertices a, b, c, d, the shape parameters are given by the cross ratios

$$z(\overline{ab}) = \frac{(b-a)(c-d)}{(a-d)(b-c)}$$
$$z(\overline{bc}) = \frac{(c-a)(b-d)}{(c-d)(b-a)}$$
$$z(\overline{cd}) = \frac{(b-c)(a-d)}{(b-d)(a-c)}$$

We will use this fact to prove that the shape parameter does not depend upon the choice of orientation of the edges.

Fact 3. The shape parameter does not depend on the orientation of the edges.

Proof. We follow Thurston's argument in [1]. Fix an edge e with shape parameter z and let a, b, c, d denote the vertices of T with e joining a and b. Since there is a unique orientation-preserving isometry of H^3 mapping a, b, c to $0, 1, \infty$ we can assume without loss of generality that three of the vertices of T lie at $a = 0$, $b = 1$, and $c = \infty$. Fix some choice of orientation of e, and consider the two faces F_1, F_2 of T sharing the edge e. The orientation on e determines orientations on each of these faces, one clockwise and the other counterclockwise. There is a unique orientation-preserving isometry of H^3 mapping F_1 onto F_2 which fixes a and b, in fact it is the Mobius transformation given by

$$\phi(w) = [w, 1; 0, d] = \frac{w(1-d)}{w-d}.$$

We note that the rotational component of ϕ has rotation angle $\operatorname{Arg} d(d-1)^{-1}$ and the translational component of ϕ has magnitude $|d(d-1)^{-1}|$. Moreover, using the cross-ratio formula for z we note that

$$z = [0, 1; \infty, d] = \frac{d}{d-1}$$

so the shape parameter is exactly the complex number whose angle is the rotational angle of ϕ and whose magnitude is the translation distance of ϕ. Now if we change the orientation of e then our orientation-preserving isometry becomes ϕ^{-1}, which possesses the same angle of rotation and translation distance as ϕ. Thus the shape parameter z is the same in either case. ■

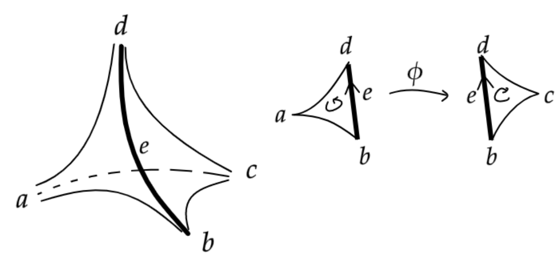

Figure 4: The orientation-preserving isometry ϕ encodes the data of the shape parameter $z(e)$.

We want to be able to glue together ideal tetrahedra in such a way that the resulting triangulation determines the structure of a hyperbolic 3-manifold (on a knot or link complement, for example). Thus, it is natural to ask: what conditions on the tetrahedra will ensure that we obtain a (complete or incomplete) hyperbolic structure? We can formulate a nice condition in terms of the shape parameters of the involved tetrahedra. Suppose that we glue ideal tetrahedra $T_1, \ldots, T_n$ along a common edge e in such a way that the triangulation produces a hyperbolic manifold structure, and let $z_1, \ldots, z_n$ denote the shape parameters corresponding to e. Without loss of generality suppose that T_1 is in standard position, i.e. three of its vertices lie at $(0, 1, t_1, \infty)$ and suppose that e is the edge joining 0 and ∞. Then using the cross ratio formula we compute

$$z_1 = \frac{t_1 - 0}{1 - 0} = t_1.$$

We want to glue the second tetrahedra along to the first along a face so that the resulting vertices are $(0, t_1, t_2, \infty)$. Using the cross ratio formula again we find that

$$z_2 = \frac{t_2 - 0}{z_1 - 0} = t_2/z_1,$$

thus $t_2 = z_1 z_2$. Repeating this procedure, we find that after the appropriate gluing, the vertices of the kth tetrahedra T_k are $(0, t_{k-1}, t_k, \infty)$ with $t_k = z_1 \cdots z_k$. Since we have glued the tetrahedra in such a way that they produce a hyperbolic manifold structure, we have some restrictions on the shape parameters: around any point on e we can find an open neighborhood which is isometric to an open ball in $\mathbb{H}^3$ – thus we must have

$$\sum_{i=1}^{n} \operatorname{Arg} z_i = \operatorname{Arg}(z_1 \cdots z_n) = \operatorname{Arg}(t_n) = 2\pi$$

and moreover, since the faces of the tetrahedra must be glued together consistently, we have $t_n = 1$ hence $z_1 \cdots z_n = 1$. Thus we have proven that the *edge gluing equations* hold for the triangulation.

Fact 4 (Edge gluing equations)**.** Suppose that we glue $T_1, \ldots, T_n$ ideal tetrahedra with shape parameters $z_1, \ldots, z_n$ to form an ideal triangulation of a hyperbolic manifold. Then

$$z_1 \cdots z_n = 1$$
$$\sum_{i=1}^{n} \operatorname{Arg} z_i = 1$$

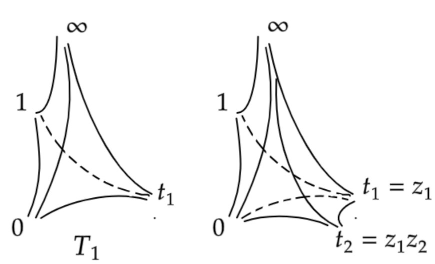

Figure 5: T_2 is glued to T_1 along a face sharing the edge between 0 and ∞.

In fact, one can prove that the converse also holds: an ideal triangulation on M which satisfies the edge gluing equations produces a hyperbolic structure on M (which may be incomplete). As a result of this "if and only if" statement, it is of interest to study the **deformation variety**, which is the set

$$\mathcal{D} = \{\mathbf{z} \in (\mathbb{C} \setminus \{0, 1, \infty\})^n : \mathbf{z} \text{ solves the edge gluing equations}\},$$

each point of which defines a hyperbolic manifold structure. As described by Weeks [2], in practice, the edge gluing equations are supplemented by cusp equations or Dehn filling equations, and then solved using Newton's method to find a hyperbolic structure.

References

[1] William Thurston, *Hyperbolic structures on 3-manifolds, I: Deformation of acylindrical manifolds*, arXiv:math/9801019, 1998

[2] Jeff Weeks, *Computation of Hyperbolic Structures in Knot Theory*, arXiv:math/0309407, 2003

[3] John Ratcliffe, *Foundations of Hyperbolic Manifolds*, Springer, 1989

[4] Nathan Dunfield's lecture notes