Topological and operator K-theory

Alex Taylor
The University of Illinois at Urbana-Champaign

Abstract

Using the K_0 group of a compact Hausdorff space to motivate the definition of the K^0 group of a C^*-algebra, we introduce operator K-theory as a non-commutative analogue of topological K-theory.

Contents

1 Topological K-theory

Given a locally compact Hausdorff space X, what are all of the possible vector bundles over X? A reasonable approach to this question is to construct a group $K^0(X)$ consisting of isomorphism classes of vector bundles over X, which might encode some information about which vector bundles X admits.

First of all, suppose that X is compact. Let $E \to X$ be a vector bundle over X. We let $[E]$ denote the equivalence class of vector bundles isomorphic to E, and consider the set of isomorphism classes:

$$V(X) = \{[E] : E = \text{vector bundle over } X\}$$

Note that $V(X)$ is a commutative monoid with respect the operation of direct sum of vector bundles over X. Namely, given two vector bundles $p : E \to X$ and $q : F \to X$ we define their direct sum by

$$E \oplus F = \bigsqcup_{x \in X} E_x \oplus F_x$$

and

$$\begin{aligned} p \oplus q &: E \oplus F \to X \\ (e, f) &\mapsto p(e) = q(f) \end{aligned}$$

so that $E \oplus F \to X$ is another vector bundle over X. The identity element of this monoid is the rank-0 trivial bundle

$$[0] = [X \times \{x\}]$$

for any $x \in X$. Unfortunately, $V(X)$ is not a group as it lacks inverses.

Example 1.

(a) When $X = \{x\}$ is a single point, we have $V(X) \simeq \mathbb{N} \cup \{0\}$.

(b) Letting $V_{\mathbb{R}}$ and $V_{\mathbb{C}}$ refer to real and complex vector bundles respectively, we have $V_{\mathbb{C}}(S^1) \simeq \mathbb{N} \cup \{0\}$ and $V_{\mathbb{R}}(S^1) \simeq (\mathbb{N} \cup \{0\}) \times \mathbb{Z}_2$.

In order to turn $V(X)$ into a group we will use the Grothendieck group construction. The general idea is to turn a commutative semigroup into a group in a "minimal" way. Given a commutative semigroup H and a subsemigroup $K \subseteq H$, define an equivalence relation on the product $H \times K$ by

$$(h_1, k_1) \sim (h_2, k_2) \iff (h_1 k_2)x = (h_2 k_1)x \text{ for some } x \in K$$

Here we are thinking of the pair $(h, k) \in H \times K$ as a fraction h/k, so that, heuristically speaking, $(h_1, k_1) \sim (h_2, k_2)$ holds if and only if $h_1/k_1 = h_2/k_2$. Then we consider the set of equivalence classes

$$[H][K]^{-1} = (H \times K)/\!\sim\, = \{[(h, k)]\}$$

and note that this is a commutative monoid with respect to the multiplication inherited from H:

$$[(h_1, k_1)] \cdot [(h_2, k_2)] = [(h_1 h_2, k_1 k_2)]$$

where the identity element is

$$1 = [(x, x)]$$

for any $x \in K$. The point of this construction is that, in this quotient space, the ordered pairs of elements of K are invertible: for any $k_1, k_2 \in K$ we have

$$[(k_1, k_2)][(k_2, k_1)] = [(k_1 k_2, k_2 k_1)] = 1$$

which is to say that $[(k_1, k_2)]^{-1} = [(k_2, k_1)]$. In essence, the commutative monoid $[H][K]^{-1}$ is obtained from H by inverting the elements of K; therefore, in the special case that $H = K$, we obtain an abelian group

$$G(H) = [H][H]^{-1}$$

called the ***Grothendieck group*** of H. We note that $G(H)$ is the "minimal" group extending the semigroup H in the sense that any homomorphism $\phi : H \to S$ of semigroups (which sends H to invertible elements of S) extends uniquely to a homomorphism $\psi : G(H) \to S$. An immediate consequence is that G is a covariant functor from the category of commutative semigroups to the category of abelian groups:

$$\begin{aligned} \{\text{commutative semigroups}\} &\xrightarrow{G} \{\text{abelian groups}\} \\ \phi : H_1 \to H_2 \quad &\mapsto \quad G(\phi) : G(H_1) \to G(H_2) \end{aligned}$$

Example 2.

(a) $G(\mathbb{N}, +) = (\mathbb{Z}, +)$

(b) $G(\mathbb{N}, \cdot) = (\mathbb{Q}_{\geq 0}, \cdot)$

Now we return to the situation where X is a compact Hausdorff space and $V(X)$ is the commutative monoid of isomorphism classes of vector bundles over X. In this case we use the Grothendieck group construction to define the group

$$K^0(X) = G(V(X)) = \{[E] - [F] : E, F = \text{vector bundles over } X\}$$

which consists of all formal differences of isomorphism classes of vector bundles over X. Notice first of all that K^0 is a *contravariant* functor from the category of compact spaces to the category of abelian groups because V is contravariant and G is covariant. First, V takes any continuous map $\phi : X \to Y$ between compact spaces and sends it to the map $\phi^* : V(Y) \to V(X)$ given by

$$\phi^* : [E \to Y] \mapsto [\phi^* E \to X]$$

where $\phi^* E \to X$ denotes the pull-back bundle induced by ϕ. Then, as ϕ^* is a morphism in the category of commutative monoids, the G functor turns it a morphism

$$G(\phi^*) : G(V(Y)) \to G(V(X))$$

in the category of abelian groups. In other words this is the morphism $K^0(\phi) : K^0(X) \to K^0(Y)$, which we shall henceforth denote by ϕ^*.

Fact 1 (Homotopy invariance)**.** *Let X, Y be compact spaces and $f, g : X \to Y$ continuous maps. If f and g are homotopic then $f^* = g^* : K^0(X) \to K^0(Y)$.*

Example 3. For any contractible space X we have $K^0(X) \simeq K^0(\{x_0\}$ by homotopy invariance, and therefore

$$K^0(X) \simeq K^0(\{x_0\}) = G(V(\{x_0\})) = G(\mathbb{N} \cup \{0\}) = \mathbb{Z}$$

In particular, for any nonempty compact space X, the function $p : X \to \{x_0\}$ induces an injective morphism $p^* : K^0(\{x_0\}) \to K^0(X)$, and therefore $K^0(X)$ always contains a copy of $\mathbb{Z}$. We define the reduced K^0-group of X by modding out by any one of these copies of $\mathbb{Z}$:

$$\widetilde{K}^0(X) = K^0(X)/\mathbb{Z}.$$

In order to get a working theory out of this K-group it's necessary to define K^0 for non-compact spaces too. For any locally compact Hausdorff space X we let X^+ denote the one-point compactification of X and then define

$$K^0(X) = \widetilde{K}^0(X^+)$$

i.e. the reduced K^0-group of the one-point compactification of X. For the remainder of this section we will assume X is a locally compact Hausdorff space. Given any closed subspace $Y \subseteq X$, the sequence

$$Y \overset{i}{\hookrightarrow} X \overset{q}{\to} X/Y$$

induces a short exact sequence of K^0-groups,

$$K^0(X/Y) \xrightarrow{q^*} K^0(X) \xrightarrow{i^*} K^0(Y)$$

given by the composition $[E] \mapsto [q^*E] \mapsto [i^*q^*E]$. Moreover, for each $n \geq 1$ we define the nth K-group as

$$K^n(X) = K^0(X \times \mathbb{R}^n)$$

and so the same argument gives another short exact sequence

$$K^n(X/Y) \xrightarrow{q^*} K^n(X) \xrightarrow{i^*} K^n(Y)$$

for every $n \geq 1$. For each n one can construct a connecting homomorphism $\delta : K^n(X) \to K^{n-1}(Y)$ and thereby get an infinite long exact sequence of K-groups which terminates in $K^0(Y)$. In fact, the sequence is actually cyclical:

Fact 2 (Bott periodicity)**.** *For any locally compact Hausdorff space Z we have*

$$K^{n+2}(Z) \simeq K^n(Z) \textit{ for every } n \geq 0$$

when complex vector bundles are considered. Furthermore, we have

$$K^{n+8}(Z) \simeq K^n(Z) \textit{ for every } n \geq 0$$

when real vector bundles are considered.

This beautiful fact reduces the study of K-groups to the study of the two groups $K^0(X)$ and $K^1(X) = K^0(X \times \mathbb{R})$ (when considering complex vector bundles over X).

2 Operator K-theory

In this section we want to explain the following common description of operator K-theory:

> *Operator K-theory is a non-commutative analogue of topological K-theory for C^*-algebras.* (Wikipedia)

We will conclude by explaining the connection between several equivalent definitions of the K^0-group of a C^*-algebra.

Suppose we have a compact Hausdorff space X. There is a one-to-one correspondence between vector bundles over X and finitely- generated projective modules over $C(X)$ given by the functor Γ sending any vector bundle E to the $C(X)$-module $\Gamma(E)$ of sections of E. Indeed, given any vector bundle $E \to X$, by Swan's theorem we can find a vector bundle $F \to X$ such that $E \oplus F \simeq X \times \mathbb{R}^n$ is a trivial bundle. Therefore

$$\begin{aligned}\Gamma(E) \oplus \Gamma(F) &\simeq \Gamma(E \oplus F) \\ &\simeq \Gamma(X \times \mathbb{R}^n) \\ &\simeq C(X)^n\end{aligned}$$

where the latter is a finitely-generated free module over $C(X)$. Thus, we have shown that $\Gamma(E)$ is a finitely-generated projective module over $C(X)$. It's not difficult to show that Γ gives a one-to-one correspondence by constructing an explicit "inverse" which associates to any such module M a vector bundle $\Psi(M) \to X$ such that $\Gamma(\Psi(M)) = M$. This observation gives us the following interpretation of the group $K^0(X)$:

Fact 3. *Let X be a compact Hausdorff space. Then $K^0(X)$ can be identified with the group of formal differences $[M]-[N]$ of isomorphism classes of finitely-generated projective modules over $C(X)$.*

Now for any commutative unital C^*-algebra A, the Gelfand-Naimark theorem gives us an isometric $*$-isomorphism $A \simeq C(X)$ for some compact space X (recall that X consists of characters on A, and the $*$-isomorphism is given by $A \to C(X)$, $a \mapsto \widehat{a}$ where $\widehat{a}(\phi) = \phi(a)$ for every character ϕ). Therefore, it makes sense to define the K_0 group of the C^*-algebra A by the prescription

$$K_0(A) = K^0(X).$$

Thus by Fact 3, we have a natural generalization to the non-commutative case: for any unital C^*-algebra A, let $K_0(A)$ be the group of formal differences of isomorphism classes $[M] - [N]$ of finitely-generated projective modules over A. In other words, if $M(A)$ denotes the monoid of isomorphism classes of finitely-generated projective A-modules, then the K_0-group of A is the Grothendieck group

$$K_0(A) = G(M(A)).$$

This is why the operator K-theory is often described as a non-commutative version of topological K-theory.

The definition of $K_0(A)$ we've taken here arises naturally from the topological K-theory group $K^0(X)$, but it's not always the most useful definition in practice. The group $K_0(A)$ can be realized in several other equivalent ways, which are often more concrete.

1. Let A be a unital C^*-algebra and define the matrix algebra of A as

   $$M_\infty(A) = \bigcup_{n \geq 1} M_n(A)$$

   and recall that two idempontents $p, q \in M_\infty(A)$ are orthogonal if $pq = qp = 0$. In this case it makes sense to define their orthogonal sum $p \oplus q \in M_\infty(A)$. We say that two idempotents p and q are equivalent if they are similar in the matrix algebra, i.e. $apa^{-1} = q$ for some invertible $a \in A$. In this case we write $p \sim q$.

   Consider the set of equivalence classes of projections in the matrix algebra:

   $$V_1 = \{[p] : p \in M_\infty(A) \text{ idempotent}\}$$

   This set is a commutative semigroup with respect to the operation

   $$[p] + [q] = [p' \oplus q']$$

   where $p' \perp q'$, $p' \sim p$ and $q' \sim q$.

2. By the Gelfand-Naimark-Segal construction we can find a faithful represention $A \to \mathbb{B}(H_A)$ of A as bounded operators on some Hilbert space H_A (this is the GNS representation). Let $\mathbb{K}(H_A)$ denote the set of compact operators on H_A, and consider the set of equivalence classes of projections

   $$V_2 = \{[P] : P \in \mathbb{K}(H_A) \text{ projection}\}$$

   Once again, this is a commutative semigroup with respect to the same operation of orthogonal sum as above.

Just like we did in section 1, we can consider the Grothendieck groups $G(V_1)$ and $G(V_2)$, which consist of formal differences of equivalence classes of idempotents and compact projections, respectively. Then

$$K_0(A) \simeq G(V_1) \simeq G(V_2)$$

so we have three equivalent realizations of the K_0-group of A. To see why these are isomorphic, let's write

$$V = \{[M] : M = \text{fgp } A\text{-module}\}$$

so that $K_0(A) = G(V)$ by definition. We have isomorphisms

$$\phi : V_1 \to V, \ \ [p] \mapsto [p(A^n)]$$

and

$$\varphi : V_2 \to V, \ \ [P] \mapsto [P(H_A)]$$

which therefore induce isomorphisms on the respective Grothendieck groups.

References

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