Lie algebras and exponential maps

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Abstract

We present a proof of the closed subgroup theorem for Lie groups using the machinery of Lie algebras and exponential maps. Along the way, we introduce all of the necessary background information about vector fields and Lie algebras.

Contents

1 Preliminaries

Let X be a smooth vector field on a smooth manifold M. Recall that we can consider X as a map $X : C^\infty(M) \to C^\infty(M)$ by defining $(Xf)(p) = X_p f$ for any $f \in C^\infty(M)$ and $p \in M$, because each tangent vector X_p is a derivation of smooth functions. Since the action of a tangent vector is locally determined, so too is the function Xf, in the sense that

$$(Xf)|_U = X(f|_U)$$

for any open subset $U \subseteq M$. As a result, we obtain a useful smoothness criterion for vector fields:

Fact 1 (Smoothness criterion for vector fields)**.** *Let M be a smooth manifold and $X : M \to TM$ any vector field on M. The following are equivalent:*

(i) X is smooth.

(ii) For every $f \in C^\infty(M)$, Xf is smooth on M.

(iii) For every open $U \subseteq M$ and every $f \in C^\infty(U)$, Xf is smooth on U.

Given a smooth map $F : M \to N$ and a vector field X on M, a ***pushfoward*** of X along F is a smooth vector field Y on N satisfying

$$Y_{F(p)} = dF_p(X_p)$$

for every $p \in M$. The pushforward of a given vector field along a given map need not exist; for example, if F is not surjective then such a vector field will not be defined outside the image of F. Or, if F is not injective, then the aforementioned formula may not produce a well-defined function. In fact, a pushforward may fail to exist even when F is a smooth bijection, because the vector field Y can fail to be smooth without the smoothness of F^{-1}.

Example 1 (Pushforward does not always exist)**.** Let $F : \mathbb{R} \to \mathbb{R}$ be the smooth bijection given by $F(x) = x^3$, and let $X = \partial/\partial t$ denote the coordinate vector field on $\mathbb{R}$. If $Y_x = y(x)\partial/\partial t|_x$ is a pushforward of X along F, then for every $x \in \mathbb{R}$ we have

$$Y_x = dF_{F^{-1}(x)}(X_{F^{-1}(x)}) = dF_{x^{1/3}}\left(\left.\frac{\partial}{\partial t}\right|_{x^{1/3}}\right) = 3x^{2/3}\left.\frac{\partial}{\partial t}\right|_{x^{1/3}}$$

which is *not* a pushforward because it's not smooth (at $x = 0$).

Despite these examples, there are some important situations where the pushforward is always uniquely defined; for example, when $F : M \to N$ is a diffeomorphism. In this case, for any smooth vector field X on M, we can define a vector field on N by

$$Y = dF \circ X \circ F^{-1} : N \xrightarrow{F^{-1}} M \xrightarrow{X} TM \xrightarrow{dF} TN$$

Evidently Y is smooth as a composition of smooth maps, and it satisfies $Y_{F(p)} = dF_p(X_p)$ by definition, so Y is a pushforward of X. For any $f \in C^\infty(N)$, the pushforward Y acts on f according to the formula

$$(Yf)_{F(p)} = dF_p(X_p)f = X_p(f \circ F)$$

or more succinctly, $Yf \circ F = X(f \circ F)$. We summarize this observation in the following fact:

Fact 2 (Pushforward along a diffeomorphism)**.** *If $F : M \to N$ is a diffeomorphism, then any vector field $X : M \to TM$ has a unique pushforward $F_*X : N \to TN$ defined by the equivalent formulas*

$$\begin{aligned} F_*X &= dF \circ X \circ F^{-1} \\ (F_*X)_{F(p)} &= dF_p(X_p) \\ (F_*X)f \circ F &= X(f \circ F) \end{aligned}$$

for every $f \in C^\infty(N)$ and $p \in M$.

Let $U \subseteq \mathbb{R}^n$ be open. A ***k-slice*** of U is a subset of the form

$$S = \{(x_1, \ldots, x_k, c_{k+1}, \ldots, c_n) \in U\} \subseteq U$$

for some constants $c_{k+1}, \ldots, c_n \in \mathbb{R}$. Thus, a k-slice is an affine subset of U homeomorphic to an open subset of $\mathbb{R}^k$. Now take a smooth n-manifold M and let $\varphi : U \subseteq M \to \mathbb{R}^n$ be a smooth chart on M. A ***k-slice*** of $U \subseteq M$ is a subset $S \subseteq U$ such that $\varphi(S)$ is a k-slice of $\varphi(U) \subseteq \mathbb{R}^n$.

We say that a subset $N \subseteq M$ satisfies the ***local k-slice condition*** if: for every $p \in N$ there exists a smooth chart $\varphi : U \subseteq M \to \mathbb{R}^n$ for M around p such that $N \cap U$ is a k-slice of U; i.e. $\varphi(N \cap U)$ is a k-slice of $\varphi(U)$. Such a chart (U, φ) is called a ***slice chart*** for N in M. Roughly speaking, the local k-slice condition says that, near any point, N is locally homeomorphic to a k-dimensional affine open subset of $\mathbb{R}^n$.

Theorem 1 (Local slice criterion)**.** *Let M be a smooth manifold. A subset $N \subseteq M$ is an embedded k-dimensional submanifold of M if and only if N satisfies the local k-slice condition.*

Recall that the differential of a smooth map can be calculated using smooth curves:

Fact 3. *Let $F : M \to N$ be a smooth map and let $p \in M$. For any $v \in T_pM$ we have*

$$dF_p(v) = (F \circ \gamma)'(0)$$

for any smooth curve $\gamma : I \to M$ satisfying $\gamma(0) = p$ and $\gamma'(0) = v$.

A Lie group is a group which is also a smooth manifold, such that the multiplication and inversion maps are smooth. For the purposes of this note we assume that the reader has a basic familiarity with Lie groups. Given a Lie group G, a ***Lie subgroup*** of G is a subgroup $H \subseteq G$ endowed with a topology and a smooth structure making it into both a Lie group and an immersed submanifold of G.

In particular, a subgroup $S \subseteq G$ which is an embedded submanifold is also an immersed submanifold, and it's easy to see that the multiplication and inversion in G restrict to smooth maps in S, so S is a Lie subgroup. Thus we have:

Fact 4 (Embedded subgroups are Lie subgroups)**.** *Let G be a Lie group and $H \subseteq G$ a subgroup that is also an embedded submanifold. Then H is a Lie subgroup of G.*

On the other hand, a Lie subgroup need not be an embedded submanifold: take an irrational winding on the 2-torus, for example. Thus, it's natural to wonder: when is a Lie subgroup an embedded submanifold? How can we characterize all of the Lie subgroups? The main goal of this note is to prove the closed subgroup theorem: a closed subgroup of a Lie group is an embedded Lie subgroup. In order to prove this theorem we will develop the machinery of Lie algebras and exponential maps.

2 Lie algebras

A ***Lie algebra*** is a real vector space $\mathfrak{g}$ equipped with an operation $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ called a ***bracket***, such that:

(i) The bracket is bilinear:

$$[au + bv, w] = a[u, w] + b[v, w]$$
$$[u, cv + dw] = c[u, v] + d[u, w]$$

for every $u, v, w \in \mathfrak{g}$ and $a, b, c, d \in \mathbb{R}$.

(ii) The bracket is antisymmetric:

$$[u, v] = -[v, u]$$

for every $u, v \in \mathfrak{g}$.

(iii) The bracket satisfies the Jacobi identity:

$$[u, [v, w]] + [w, [u, v]] + [v, [w, u]] = 0$$

for every $u, v, w \in \mathfrak{g}$.

Let $\mathfrak{g}$ be a Lie algebra. We establish the following basic definitions:

- A ***Lie subalgebra*** of $\mathfrak{g}$ is a linear subspace $\mathfrak{h} \subseteq \mathfrak{g}$ which is closed under brackets; i.e. $\mathfrak{h}$ is itself a Lie algebra with respect to (the restriction of) the same bracket.
- A linear map $A : \mathfrak{g} \to \mathfrak{h}$ between Lie algebras is called a ***Lie algebra homomorphism*** if A preserves brackets; that is,

$$A[u, v] = [Au, Av]$$

for every $u, v \in \mathfrak{g}$.
- A bijective Lie algebra homomorphism is called a ***Lie algebra isomorphism***.

Example 2 (Lie algebras)**.**

(a) The vector space $M(n, \mathbb{R})$ of all $n \times n$ real matrices is an n^2-dimensional Lie algebra with respect to the commutator bracket $[A, B] = AB - BA$.

(b) Similarly, the vector space $M(n, \mathbb{C})$ of all $n \times n$ complex matrices is a Lie algebra with respect to the commutator bracket.

(c) In general, for any finite-dimensional vector space V, the vector space $\mathrm{Hom}(V)$ consisting of all linear maps $V \to V$ is a Lie algebra with respect to the commutator bracket $[A, B] = A \circ B - B \circ A$.

(d) $\mathbb{R}^3$ is a Lie algebra with bracket given by the cross product of vectors: $[u, v] = u \times v$.

(e) Given a smooth manifold M, the vector space $\Gamma(TM)$ of smooth vector fields on M is a Lie algebra with respect to the bracket $[X, Y] = XY - YX$. Recall that vector fields on M act as maps $C^\infty(M) \to C^\infty(M)$, so here $[X, Y]$ is the vector field given by

$$[X, Y]f = X(Yf) - Y(Xf)$$

for every $f \in C^\infty M$.

(f) As we shall see in the next section, given a Lie group G, the subspace of $\Gamma(TG)$ consisting of all *left-invariant* vector fields on G is a Lie algebra with respect to the same bracket $[X, Y] = XY - YX$. This is the Lie algebra associated with the Lie group G.

Many standard linear algebra facts carry over to the setting of Lie algebras.

Fact 5. *Let* $A : \mathfrak{g} \to \mathfrak{h}$ *be a Lie algebra homomorphism. Then* $\ker A \subseteq \mathfrak{g}$ *and* $\operatorname{im} A \subseteq \mathfrak{h}$ *are Lie subalgebras.*

Proof. The kernel and image of A are linear subspaces for algebraic reasons, so it suffices to check that they are closed under the brackets on $\mathfrak{g}$ and $\mathfrak{h}$, respectively. For any $u, v \in \ker A$ we have

$$A[u, v] = [Au, Av] = [0, 0] = 0$$

so $[u, v] \in \ker A$ and the kernel is closed under brackets. Similarly, for any $u, v \in \mathfrak{g}$ the equation $[Au, Av] = A[u, v]$ implies that $[Au, Av] \in \operatorname{im} A$; hence the image is closed under brackets. ■

3 The Lie algebra of a Lie group

One of the most important examples of a Lie algebra is the vector space of smooth vector fields on a smooth manifold; especially because every Lie group is associated with a particular Lie algebra consisting of left-invariant vector fields. In this section we will explain how this relationship works.

Let X be a smooth vector field on a smooth manifold M, considered as a map $X : C^\infty(M) \to C^\infty(M)$. This map is linear over $\mathbb{R}$, and it satisfies the product rule

$$X(fg)(p) = g(p)X_p(f) + f(p)X_p(g)$$

i.e. $X(fg) = gXf + fXg$. These properties follow immediately from the corresponding properties for derivations, and in fact, smooth vector fields are completely characterized as maps $C^\infty(M) \to C^\infty(M)$ satisfying these two properties.

Let M be a smooth manifold and let X, Y be smooth vector fields on M. Then we can apply both X and Y to any function $f \in C^\infty(M)$ to get a map $f \mapsto YXf$, however this map does *not* satisfy the product rule, so YX does not define a vector field on M. Instead, we can combine X and Y by essentially taking their commutator: $[X, Y] = XY - YX$. This does define a smooth vector field on M called the ***Lie bracket*** of X and Y.

Fact 6. *For any smooth vector fields* X *and* Y *on* M*, the Lie bracket* $[X, Y] = XY - YX$ *is a smooth vector field on* M*.*

Proof. Based on the preceding discussion, it suffices to show that $[X, Y]$ is a linear map $C^\infty(M) \to C^\infty(M)$ satisfying the product rule. Linearity is pretty clear:

$$\begin{aligned}
[X, Y](af + bg) &= X(Y(af + bg)) - Y(X(af + bg)) \\
&= X(aYf + bYg) - Y(aXf + bXg) \\
&= aXYf + bXYg - aYXf - bYXg \\
&= a(XY - YX)f + b(XY - YX)g \\
&= a[X, Y]f + b[X, Y]g
\end{aligned}$$

for any $f, g \in C^\infty(M)$ and $a, b \in \mathbb{R}$. Now we check the product rule:

$$\begin{aligned}
[X,Y](fg) &= X(Y(fg)) - Y(X(fg)) \\
&= X(fY(g) + gY(f)) - Y(fX(g) + gX(f)) \\
&= fXY(g) + Y(g)X(f) + gXY(f) + X(g)Y(f) \\
&\quad - [fYX(g) + Y(f)X(g)] - [gYX(f) + X(f)Y(g)] \\
&= fXY(g) - fYX(g) + gXY(f) - gYX(f) \\
&= f(XY - YX)g + g(XY - YX)f \\
&= f[X,Y]g + g[X,Y]f
\end{aligned}$$

as desired. ∎

Fact 7 (Properties of the Lie bracket)**.** *The Lie bracket on the space of smooth vector fields on a smooth manifold M satisfies:*

(i) Bilinearity:

$$\begin{aligned}
[aX + bY, Z] &= a[X,Z] + b[Y,Z] \\
[X, cY + dZ] &= c[X,Y] + d[X,Z]
\end{aligned}$$

for every X, Y, and $Z \in \Gamma(TM)$ and $a, b, c, d \in \mathbb{R}$.

(ii) Antisymmetry:

$$[X,Y] = -[Y,X]$$

for every $X, Y \in \Gamma(TM)$.

(iii) The Jacobi identity:

$$[X,[Y,Z]] + [Z,[X,Y]] + [Y,[Z,X]] = 0$$

for every X, Y, and $Z \in \Gamma(TM)$.

(iv) For every $f, g \in C^\infty(M)$:

$$[fX, gY] = fg[X,Y] + (fXg)Y - (gYf)X.$$

Proof. Bilinearity and antisymmetry follow immediately from the definition of the Lie bracket. For the Jacobi identity, a direct computation shows that for any $f \in C^\infty(M)$,

$$\begin{aligned}
[X,[Y,Z]]f + [Y,[Z,X]]f + [Z,[X,Y]]f &= X[Y,Z]f - [Y,Z]Xf + Y[Z,X]f \\
&\quad - [Z,X]Yf + Z[X,Y]f - [X,Y]Zf \\
&= X(YZ - ZY)f - (YZ - ZY)Xf \\
&\quad + Y(ZX - XZ)f - (ZX - XZ)Yf \\
&\quad + Z(XY - YX)f - (XY - YX)Zf \\
&= 0
\end{aligned}$$

after expanding the last line and seeing that the terms cancel in pairs. For property (iv), we have for any $h \in C^\infty(M)$,

$$\begin{aligned}
[fX, gY]h &= (fX)(gY)h - (gY)(fX)h \\
&= fX(gYh) - gY(fXh) \\
&= f\big(gX(Yh) + (Yh)(Xg)\big) - g\big(fY(Xh) + (Xh)(Yf)\big) \\
&= f(gX)(Yh) + f(Yh)(Xg) - g(fY)(Xh) - g(Xh)(Yf) \\
&= fg[X,Y]h + (fXg)Yh - (gYf)Xh
\end{aligned}$$

∎

As a result of Facts 6 and 7, we have shown that the vector space $\Gamma(TM)$ of smooth vector fields on M is a Lie algebra with respect to the Lie bracket. Here is one more useful fact: the Lie bracket is preserved by pushforwards.

Fact 8 (Pushforward invariance of the Lie bracket)**.** *Let $F : M \to N$ be a smooth map, and let X_1 and X_2 be smooth vector fields with well-defined pushforward F_*X_1 and F_*X_2 along F. Then the pushforward of $[X_1, X_2]$ is well-defined and*

$$F_*[X_1, X_2] = [F_*X_1, F_*X_2].$$

Proof. Write $Y_1 = F_*X_1$ and $Y_2 = F_*X_2$. Using Fact 2, for any $f \in C^\infty(N)$ we calculate

$$\begin{aligned} X_1X_2(f \circ F) &= X_1(Y_2 f \circ F) = (Y_1Y_2 f) \circ F \\ X_2X_1(f \circ F) &= X_2(Y_1 f \circ F) = (Y_2Y_1 f) \circ F \end{aligned}$$

and therefore

$$\begin{aligned} (X_1X_2 - X_2X_1)(f \circ F) &= (Y_1Y_2 f) \circ F - (Y_2Y_1 f) \circ F \\ &= ([Y_1, Y_2] f) \circ F \end{aligned}$$

which means that

$$F_*[X_1, X_2] = [Y_1, Y_2] = [F_*X_1, F_*X_2],$$

as desired. ∎

In this note we will focus on Lie groups with vector fields satisfying an additional left-invariance condition. Let G be a Lie group, and for any $g \in G$ let $L_g : G \to G$ denote the left-multiplication diffeomorphism. A smooth vector field X on G is ***left-invariant*** if it satisfies

$$(L_g)_*X = X$$

for every $g \in G$. In other words, this means that

$$d(L_g)_{g'}(X_{g'}) = X_{gg'}$$

for every $g, g' \in G$. Notice that a left-invariant vector field X on G is completely determined by its value at the identity element $e \in G$: for any $g \in G$ we have

$$X_g = X_{ge} = d(L_g)_e(X_e).$$

Furthermore, note that the property of being left-invariant is preserved by linear combinations because if X and Y are left-invariant vector fields on G then we have

$$\begin{aligned} (L_g)_*(aX + Y)f &= (aX + Y)(f \circ L_g) \circ L_g^{-1} \\ &= aX(f \circ L_g) \circ L_g^{-1} + Y(f \circ L_g) \circ L_g^{-1} \\ &= a(L_g)_*(X)f + (L_g)_*(Y)f \end{aligned}$$

for any $f \in C^\infty(M)$, $a \in \mathbb{R}$ and $g \in G$. Left-invariance is also preserved by Lie brackets because by Fact 8 we have

$$(L_g)_*[X, Y] = [(L_g)_*X, (L_g)_*Y] = [X, Y]$$

which means that $[X,Y]$ is left-invariant. In summary, we have shown that the space of smooth left-invariant vector fields on G is a Lie subalgebra of $\Gamma(TG)$; in particular it is a Lie algebra with respect to the Lie bracket. We denote this Lie algebra by the symbol $\mathrm{Lie}(G)$ and call it the ***Lie algebra of the Lie group*** G.

Since a left-invariant vector field on a Lie group G is determined by its value at the identity, it is reasonable to expect that there should be a one-to-one correspondence between the space $\mathrm{Lie}(G)$ of left-invariant vector fields, and the space T_eG of tangent vectors at the identity. This observation leads to the following fact.

Fact 9. *Let G be a Lie group. The evaluation map*

$$\varepsilon : \mathrm{Lie}(G) \to T_eG, \;\; \varepsilon(X) = X_e$$

is a linear isomorphism. Thus, $\mathrm{Lie}(G)$ *is a finite-dimensional vector space with* $\dim \mathrm{Lie}(G) = \dim G$.

Proof.

- ε is linear: for $X, Y \in \mathrm{Lie}(G)$ and $a, b \in \mathbb{R}$ we have

$$\varepsilon(aX + bY) = (aX + bY)_e = aX_e + bY_e = a\varepsilon(X) + b\varepsilon(Y).$$

- ε is injective. Suppose that $\varepsilon(X) = X_e = 0$. Since X is left-invariant, we have

$$X_g = X_{ge} = d(L_g)_e(X_e) = d(L_g)_e(0) = 0$$

for every $g \in G$, so $X = 0$. Thus $\ker \varepsilon = 0$ and ε is injective.

- ε is surjective. Let $v \in T_eG$ be arbitrary. We want to find a left-invariant smooth vector field X such that $\varepsilon(X) = X_e = v$; i.e. X yields the vector v at the identity. If X satisfies this condition, then by left-invariance we must have

$$X_g = X_{ge} = d(L_g)_e(X_e) = d(L_g)_e(v)$$

for every $g \in G$. Thus we define a vector field $v^L : G \to TG$ by the formula $v^L(g) = d(L_g)_e(v)$. We need to show that $v^L \in \mathrm{Lie}(G)$, i.e. that it is smooth and left-invariant. In order to show that v^L is smooth, by Fact 1 it suffices to show that $v^L f$ is a smooth function on G for any $f \in C^\infty(G)$. We have

$$(v^L f)(g) = v^L(g)(f) = d(L_g)_e(v)(f) = v(f \circ L_g).$$

Choose any smooth curve $\gamma : (-\delta, \delta) \to G$ with $\gamma(O) = g$ and $\gamma'(0) = v$. We compute

$$\begin{aligned}(v^L f)(g) &= v(f \circ L_g) \\ &= \gamma'(0)(f \circ L_g) \\ &= (f \circ L_g \circ \gamma)'(0) \\ &= \left.\frac{d}{dt}\right|_{t=0} (f \circ L_g \circ \gamma)(t).\end{aligned}$$

Define $\alpha : (-\delta, \delta) \times G \to \mathbb{R}$ by $\alpha(t, g) = (f \circ L_g \circ \gamma)(t)$. Then α is smooth in both t and g, and

$$v^L(f)(g) = \left.\frac{d}{dt}\right|_{t=0} (f \circ L_g \circ \gamma)(t) = \frac{\partial \alpha}{\partial t}(0, g)$$

which is a smooth function of g. Hence $v^L f$ is smooth on G and v^L is a smooth vector field. Finally, we check that v^L is left-invariant. For any $g, h \in G$ we have

$$\begin{aligned} d(L_g)_h(v^L(h)) &= d(L_g)_h(d(L_h)_e(v)) \\ &= d(L_g \circ L_h)_e(v) \\ &= d(L_{gh})_e(v) \\ &= v^L(gh). \end{aligned}$$

Hence $v^L \in \mathrm{Lie}(G)$ satisfies $\varepsilon(v^L) = v^L(e) = v$ and ε is surjective.

■

Following the notation of Fact 9, for any tangent vector $v \in T_eG$ we will let v^L denote the unique left-invariant smooth vector field on G whose value at the identity is v.

Note that Fact 9 also shows that every left-invariant vector field on a Lie group is smooth: if X is a left-invariant vector field on a Lie group G, then evidently $X = (X_e)^L$ and the latter is a smooth vector field by Fact 9. Hence the modifier "smooth" in the phrase "smooth left-invariant vector field" is redundant.

Now that we have a linear isomorphism between the Lie algebra $\mathrm{Lie}(G)$ and the vector space T_eG, we can endow the latter with a canonical Lie algebra structure: there is a unique bracket on T_eG for which the evaluation map ϵ is an isomorphism of Lie algebras. Precisely, for any $v, w \in T_eG$ we define the bracket

$$[v, w] = \epsilon[\epsilon^{-1}v, \epsilon^{-1}w].$$

We can go further still: for any $g \in G$ we have a natural isomorphism $d(L_g)_e : T_eG \to T_gG$, and so we can endow T_gG with a Lie algebra structure by selecting the unique bracket for which the linear isomorphism $d(L_g)_e$ becomes a Lie algebra isomorphism. Namely, for any $\sigma, \tau \in T_gG$ we could define the bracket

$$[\sigma, \tau] = d(L_g)_e[d(L_g)_e^{-1}(\sigma), d(L_g)_e^{-1}(\tau)]$$

and thus every tangent space T_gG has a Lie algebra structure such that $\mathrm{Lie}(G) \simeq T_gG$ as Lie algebras. Since the manifold G has some additional algebraic structure, it makes sense that its tangent spaces would possess some additional algebraic structure too. In any case, it makes sense to focus on T_eG in particular because the identity element is distinguished, and because any left-invariant vector field on G is determined by its value at $e \in G$.

Example 3.

(a) Let $G = \mathbb{R}^n$ with respect to vector addition. Left translation by $b \in \mathbb{R}^n$ is given by the affine transformation $L_b(x) = b + x$, whose differential $d(L_b)$ is represented by the identity matrix in standard coordinates. Thus, a vector field X on $\mathbb{R}^n$ is left-invariant if and only if, for every $a, b \in \mathbb{R}^n$,

$$d(L_b)_a(X_a) = X_{a+b} \iff X_a = X_{a+b}$$

and this holds if and only if the components X^i of X are constant. Since the Lie bracket of two constant vector fields is zero, $\mathrm{Lie}(\mathbb{R}^n)$ is just $\mathbb{R}^n$ itself equipped with the zero bracket.

(b) Let $G = S^1$ with respect to multiplication in $\mathbb{C}$. Left translation in S^1 has the local coordinate representation $\theta \mapsto \theta + c$ for some fixed $c \in \mathbb{R}$. The differential of this map is represented by the 1×1 identity matrix, so the coordinate vector field $d/d\theta$ on S^1 is left-invariant. Thus $d/d\theta$ is a basis for the Lie algebra of S^1 (equipped with the zero bracket) and we conclude that $\mathrm{Lie}(S^1) = \mathbb{R}$.

(c) Let $G = \mathbb{T}^n = S^1 \times \cdots \times S^1$ denote the n-dimensional torus. Following the same logic as above, the coordinate vector fields $\{\partial/\partial\theta_1, \ldots, \partial/\partial\theta_n\}$ form a basis for the Lie algebra of $\mathbb{T}^n$ equipped with the zero bracket, hence $\mathrm{Lie}(\mathbb{T}^n) = \mathbb{R}^n$.

Any Lie group homomorphism $F : G \to H$ induces a Lie algebra homomorphism $F_* : \mathrm{Lie}(G) \to \mathrm{Lie}(H)$ defined by the formula

$$F_*(X) = dF_e(X_e)^L = \varepsilon^{-1}(dF_e(X_e)).$$

Note that $F_* = \varepsilon^{-1} \circ dF_e \circ \varepsilon$ if we denote the evaluation maps for both $\mathrm{Lie}(G)$ and $\mathrm{Lie}(H)$ by the same symbol ε. To see why this defines a Lie algebra homomorphism, notice that $F_*(X)$ really is a pushforward of X along F: since F is a group homomorphism we have

$$F(gg') = F(g)F(g') \Rightarrow F(L_g(g')) = L_{F(g)}(F(g'))$$

for every $g, g' \in G$, so $F \circ L_g = L_{F(g)} \circ F$. Differentiating both sides of this equation, we obtain

$$dF \circ d(L_g) = d(L_{F(g)}) \circ dF.$$

For any $X \in \mathrm{Lie}(G)$, let $Y = dF_e(X_e)^L \in \mathrm{Lie}(H)$ so that $Y_e = dF_e(X_e)$. It follows that

$$\begin{aligned} dF_g(X_g) &= dF_g(d(L_g)_e(X_e)) \\ &= d(L_{F(g)})_e(dF_e(X_e)) \\ &= d(L_{F(g)})_e(Y_e) \\ &= Y_{F(g)}, \end{aligned}$$

and this means exactly that $Y = F_*(X)$ is a pushforward of X along F. Hence F_* preserves Lie brackets by Fact 8, and F_* is clearly linear as a composition of linear maps. We conclude that F_* is a well-defined Lie algebra homomorphism, called the ***induced Lie algebra homomorphism***. This concept is important because it allows us to pass information between Lie groups and their associated Lie algebras.

Fact 10 (Properties of induced homomorphisms)**.**

(i) The identity map $\mathrm{id}_G : G \to G$ *induces the identity map* $\mathrm{id}_{\mathrm{Lie}(G)} = (\mathrm{id}_G)_* : \mathrm{Lie}(G) \to \mathrm{Lie}(G)$.

(ii) If $F_1 : G \to H$ *and* $F_2 : H \to K$ *are Lie group homomorphisms, then* $(F_2 \circ F_1)_* = (F_2)_* \circ (F_1)_* : \mathrm{Lie}(G) \to \mathrm{Lie}(K)$.

(iii) An isomorphism of Lie groups $F : G \to H$ *induces an isomorphism of Lie algebras* $F_* : \mathrm{Lie}(G) \to \mathrm{Lie}(H)$. *Thus,* $G \simeq H$ *implies that* $\mathrm{Lie}(G) \simeq \mathrm{Lie}(H)$.

Proof. (i) We have $(\mathrm{id}_G)_* = \epsilon^{-1} \circ d(\mathrm{id}_G)_e \circ \epsilon = \epsilon^{-1} \circ \epsilon = \mathrm{id}_{\mathrm{Lie}(G)}$ since the differential of the identity map on G is the identity map on T_eG.

(ii) For the sake of precision let us denote the evaluation maps for G, H and K by $\varepsilon_G, \varepsilon_H$ and ε_K. We have by definition,

$$\begin{aligned}(F_2)_* \circ (F_1)_* &= (\varepsilon_K^{-1} \circ d(F_2)_e \circ \varepsilon_H) \circ (\varepsilon_H^{-1} \circ d(F_1)_e \circ \varepsilon_G) \\ &= \varepsilon_K^{-1} \circ d(F_2)_e \circ d(F_1)_e \circ \varepsilon_G \\ &= \varepsilon_K^{-1} \circ d(F_2 \circ F_1)_e \circ \varepsilon_G \\ &= (F_2 \circ F_1)_*\end{aligned}$$

(iii) This follows immediately from (i) and (ii), because if $F : G \to H$ is an isomorphism then $F_* \circ F_*^{-1} = (F \circ F^{-1})_* = (\mathrm{id}_H)_* = \mathrm{id}_{\mathrm{Lie}(H)}$ and similarly $F_*^{-1} \circ F_* = \mathrm{id}_{\mathrm{Lie}(G)}$, so F_* is a Lie algebra isomorphism.

■

In other words, Fact 10 says that we have a functor from the category of Lie groups into the category of Lie algebras, sending any Lie group to its associated Lie algebra $\mathrm{Lie}(G)$, and any morphism F of Lie groups to its induced morphism F_* of Lie algebras.

Let G be a Lie group and $H \subseteq G$ a Lie subgroup. We might expect $\mathrm{Lie}(H)$ to be a Lie subalgebra of $\mathrm{Lie}(G)$, but the elements of $\mathrm{Lie}(H)$ are vector fields on H and not G, so technically speaking they are not elements of $\mathrm{Lie}(G)$. Nonetheless, we can still identify $\mathrm{Lie}(H)$ with a Lie subalgebra of $\mathrm{Lie}(G)$ in a canonical way.

Fact 11. *Let $H \subseteq G$ be a Lie subgroup and $i : H \hookrightarrow G$ the inclusion. Then* $\mathrm{Lie}(H)$ *is isomorphic to the Lie subalgebra of* $\mathrm{Lie}(G)$ *defined as*

$$\mathfrak{h} = i_*(\mathrm{Lie}(H)) = \{X \in \mathrm{Lie}(G) : X_e \in T_eH\}$$

Proof. The inclusion $i : H \hookrightarrow G$ is a Lie group homomorphism inducing a Lie algebra homomorphism $i_* : \mathrm{Lie}(H) \to \mathrm{Lie}(G)$, so the image $\mathfrak{h} = i_*(\mathrm{Lie}(H))$ is a Lie subalgebra of $\mathrm{Lie}(G)$ by Fact 5. Note that the differential $di_e : T_eH \hookrightarrow T_eG$ is injective, so $i_* = \varepsilon^{-1} \circ di_e \circ \varepsilon$ is injective as a composition of injections. Thus $i_* : \mathrm{Lie}(H) \to \mathfrak{h}$ is an isomorphism onto its image and we conclude that $\mathfrak{h} \simeq \mathrm{Lie}(H)$ as Lie algebras.

We also note that $\mathfrak{h}$ consists of those left-invariant vector fields on G tangent to H at e, because $Y \in \mathfrak{h}$ if and only if $Y = i_*(X)$ for some $X \in \mathrm{Lie}(H)$, if and only if $Y_e = di_e(X_e)$. This is equivalent to saying that $Y_e \in T_eH$ since $di_e : T_eH \hookrightarrow T_eG$ is the inclusion of tangent spaces. ■

Fact 12. *Let $F : G \to H$ be a Lie group homomorphism and $F_* : \mathrm{Lie}(G) \to \mathrm{Lie}(H)$ its induced Lie algebra homomorphism. Then*

(i) $\ker F_* = \mathrm{Lie}(\ker F)$ *(under the identification of Fact 11).*

(ii) If F is an immersion, then $\mathrm{im}\, F_* = \mathrm{Lie}(\mathrm{im}\, F)$ *(under the identification of Fact 11).*

(iii) If F is a local diffeomorphism, then F_* is an isomorphism of Lie algebras.

Proof. (i) Let $i : \ker F \hookrightarrow G$ denote the inclusion. We show that $\ker F_* = i_*(\mathrm{Lie}(\ker F))$. Let $X \in \mathrm{Lie}(G)$ be arbitrary, then

$$\begin{aligned} X \in \ker F_* &\iff Y = F_*(X) = 0 \\ &\iff Y_e = dF_e(X_e) = 0 \\ &\iff X_e \in \ker dF_e \end{aligned}$$

But $\ker dF_e = T_e \ker F$, so $X \in \ker F_*$ holds if and only if $X_e \in T_e \ker F$, which is to say that $\ker F_* = i_*(\mathrm{Lie}(\ker F))$ as desired.

(ii) Let $i : \mathrm{im}\, F \hookrightarrow H$ denote the inclusion. We show that $\mathrm{im}\, F_* = i_*(\mathrm{Lie}(\mathrm{im}\, F))$. Let $Y \in \mathrm{Lie}(H)$ be arbitrary, then

$$\begin{aligned} Y \in \mathrm{im}\, F_* &\iff Y = F_*(X) \text{ for some } X \in \mathrm{Lie}(G) \\ &\iff Y_e = dF_e(X_e) \\ &\iff Y_e \in dF_e(T_e G) \end{aligned}$$

On the other hand, since F is an immersion we have that $dF_e(T_e G) = T_e \,\mathrm{im}\, F$, so $Y \in \mathrm{im}\, F_*$ holds if and only if $Y_e \in T_e \,\mathrm{im}\, F$. Thus $\mathrm{im}\, F_* = i_*(\mathrm{Lie}(\mathrm{im}\, F))$ as desired.

(iii) If F is a local diffeomorphism then $dF_e : T_e G \to T_e H$ is a linear isomorphism, so $F_* = \varepsilon^{-1} \circ dF_e \circ \varepsilon$ is an isomorphism.

■

4 Lie algebras of matrix groups

Some of the most important examples of Lie groups are the matrix groups such as $M(n, \mathbb{R}), \mathrm{GL}(n, \mathbb{R}), O(n), U(n)$, and so on. In this section we will describe the Lie algebras associated with some of these Lie groups.

First of all, let's consider the general linear group $\mathrm{GL}(n, \mathbb{R})$. Since $\mathrm{GL}(n, \mathbb{R})$ is an open submanifold of $M(n, \mathbb{R})$, we have a linear isomorphism $T_I \,\mathrm{GL}(n, \mathbb{R}) \simeq M(n, \mathbb{R})$; furthermore, by Fact 9 the evaluation map provides an isomorphism $\mathrm{Lie}(\mathrm{GL}(n, \mathbb{R})) \simeq T_I \,\mathrm{GL}(n, \mathbb{R})$. By combining these isomorphisms we find that

$$\mathrm{Lie}(\mathrm{GL}(n, \mathbb{R})) \simeq T_I \,\mathrm{GL}(n, \mathbb{R}) \simeq M(n, \mathbb{R})$$

On the other hand, $\mathrm{Lie}(\mathrm{GL}(n, \mathbb{R}))$ and $M(n, \mathbb{R})$ are both Lie algebras equipped with their own brackets, the former is the Lie bracket on vector fields, and the latter is the commutator bracket on matrices. A natural question arises: what is the relationship between these two Lie algebra structures? In fact, the aforementioned vector space isomorphism $\mathrm{Lie}(\mathrm{GL}(n, \mathbb{R})) \simeq M(n, \mathbb{R})$ is actually an isomorphism of Lie algebras. Before proving this assertion, let's recall some basic information about the tangent space of $\mathrm{GL}(n, \mathbb{R})$:

1. Since $M(n, \mathbb{R})$ is a vector space we have a natural isomorphism

$$\begin{aligned} T_I M(n, \mathbb{R}) &\to M(n, \mathbb{R}) \\ \left.\frac{\partial}{\partial x_{ij}}\right|_I &\mapsto E_{ij} \end{aligned}$$

where $E_{ij} \in M(n,\mathbb{R})$ is the matrix whose (i,j)-entry is 1 and all others are zero, and $\partial/\partial x_{ij}|_I$ denotes the directional derivative at I in the direction of E_{ij}; i.e.

$$\frac{\partial}{\partial x_{ij}}\bigg|_I (f) = \frac{d}{dt}\bigg|_{t=0} f(I + tE_{ij})$$

for every smooth $f : M(n,\mathbb{R}) \to \mathbb{R}$.

2. Since $\mathrm{GL}(n,\mathbb{R})$ is an open submanifold of $M(n,\mathbb{R})$, we have an isomorphism induced by the inclusion, hence

$$T_I \,\mathrm{GL}(n,\mathbb{R}) = \operatorname{span}\left\{ \frac{\partial}{\partial x_{ij}}\bigg|_I \right\}$$

3. Combining the previous two remarks, the isomorphism $T_I \,\mathrm{GL}(n,\mathbb{R}) \simeq M(n,\mathbb{R})$ is given by

$$\sum_{i,j} A_{ij} \frac{\partial}{\partial x_{ij}}\bigg|_I \mapsto (A_{ij})$$

Fact 13 (Lie algebra of general linear group)**.** *The composition of natural isomorphisms*

$$\mathrm{Lie}(\mathrm{GL}(n,\mathbb{R})) \simeq T_I \,\mathrm{GL}(n,\mathbb{R}) \simeq M(n,\mathbb{R})$$

yields a Lie algebra isomorphism.

Proof. Under the right-hand isomorphism, any matrix $A = (A_{ij}) \in M(n,\mathbb{R})$ uniquely determines a tangent vector $\sum_{i,j} A_{ij}\partial/\partial x_{ij}|_I \in T_I \,\mathrm{GL}(n,\mathbb{R})$, which in turn (under the left-hand isomorphism) determines a left-invariant vector field $A^L \in \mathrm{Lie}(\mathrm{GL}(n,\mathbb{R}))$ defined by

$$A^L|_Y = d(L_Y)_I \left(\sum_{i,j} A_{ij} \frac{\partial}{\partial x_{ij}}\bigg|_I \right) = d(L_Y)_I(A)$$

for any $Y \in \mathrm{GL}(n,\mathbb{R})$. Thus we want to show that the vector space isomorphism $M(n,\mathbb{R}) \to \mathrm{Lie}(\mathrm{GL}(n,\mathbb{R}))$ given by $A \mapsto A^L$ is actually a Lie algebra isomorphism; in other words we must show that $[A,B]^L = [A^L, B^L]$ for any $A, B \in M(n,\mathbb{R})$, where the left-hand bracket is the commutator of matrices and the right-hand bracket is the Lie bracket of vector fields.

Notice that the left multiplication map $L_Y : \mathrm{GL}(n,\mathbb{R}) \to \mathrm{GL}(n,\mathbb{R})$ given by $B \mapsto YB$ is (the restriction of) a linear map, so its differential is simply $d(L_Y)_I = L_Y : M(n,\mathbb{R}) \to M(n,\mathbb{R})$. As a result, we have

$$\begin{aligned} A^L|_Y &= d(L_Y)_I \left(\sum_{i,j} A_{ij} \frac{\partial}{\partial x_{ij}}\bigg|_I \right) \\ &= \sum_{i,j} \left(\sum_k Y_{ik} A_{kj} \right) \frac{\partial}{\partial x_{ij}}\bigg|_Y \\ &= YA \end{aligned}$$

Fix any two arbitrary matrices $A, B \in M(n,\mathbb{R})$, and fix the following shorthand: we define functions $\alpha^{ij} : M(n,\mathbb{R}) \to \mathbb{R}$ and $\beta^{pq} : M(n,\mathbb{R}) \to \mathbb{R}$ by

$$\begin{aligned} \alpha^{ij}(Y) &= (YA)_{ij} = \ (i,j)\text{-entry of } YA \\ \beta^{pq}(Y) &= (YB)_{pq} = \ (p,q)\text{-entry of } YB \end{aligned}$$

for any $Y \in M(n,\mathbb{R})$. These are precisely the component functions for A^L and B^L: we have

$$A^L = \sum_{i,j} \alpha^{ij} \frac{\partial}{\partial x_{ij}} \quad \text{and} \quad B^L = \sum_{p,q} \beta^{pq} \frac{\partial}{\partial x_{pq}}$$

Thus we calculate

$$\begin{aligned} [A^L, B^L]\big|_I &= A^L|_I B^L - B^L|_I A^L \\ &= \sum_{i,j}\sum_{p,q} A_{ij} \frac{\partial \beta^{pq}}{\partial x_{ij}}(I) \frac{\partial}{\partial x_{pq}} - \sum_{p,q}\sum_{i,j} B_{pq} \frac{\partial \alpha^{ij}}{\partial x_{pq}}(I) \frac{\partial}{\partial x_{ij}} \end{aligned} \tag{1}$$

Now we need to determine the partial derivatives of the component functions α^{ij} and β^{pq}. Note that

$$\begin{aligned} \frac{\partial \beta^{pq}}{\partial x_{ij}}(I) &= \frac{d}{dt}\bigg|_{t=0} \beta^{pq}(I + tE_{ij}) \\ &= \frac{d}{dt}\bigg|_{t=0} ((I + tE_{ij})B)_{pq} \\ &= \frac{d}{dt}\bigg|_{t=0} (B + tE_{ij}B)_{pq} \\ &= \begin{cases} \frac{d}{dt}\big|_{t=0}(B_{pq} + tB_{jq}) & \text{if } p = i \\ 0 & \text{if } p \neq i \end{cases} \\ &= \begin{cases} B_{jq} & \text{if } p = i \\ 0 & \text{if } p \neq i \end{cases} \end{aligned}$$

where we have used the fact that $(E_{ij}B)_{pq} = B_{jq}$ because the ith row of $E_{ij}B$ is the jth row of B, and all other entries are zero. Similarly,

$$\frac{\partial \alpha^{ij}}{\partial x_{pq}}(I) = \begin{cases} A_{qj} & \text{if } i = p \\ 0 & \text{if } i \neq p \end{cases}$$

Therefore, by plugging these values into the expression (1) and swapping the indices j and q in the first sum, we have

$$\begin{aligned} [A^L, B^L]\big|_I &= \sum_{i,j}\sum_{q} A_{ij}B_{jq} \frac{\partial}{\partial x_{iq}}\bigg|_I - \sum_{q}\sum_{i,j} B_{iq}A_{qj} \frac{\partial}{\partial x_{ij}}\bigg|_I \\ &= \sum_{i,j}\left(\sum_q A_{iq}B_{qj} - B_{iq}A_{qj}\right) \frac{\partial}{\partial x_{ij}}\bigg|_I \\ &= \sum_{i,j}(AB - BA)_{ij} \frac{\partial}{\partial x_{ij}}\bigg|_I \\ &= [A, B]^L\big|_I \end{aligned}$$

The result now follows from the fact that any left-invariant vector field is determined by its value at the identity; we have $[A^L, B^L] = [A, B]^L$ and this gives us an isomorphism of Lie algebras. ■

In light of the Lie algebra isomorphism provided to us by Fact 13, we use the symbol $\mathfrak{gl}(n,\mathbb{R})$ to denote both the Lie algebra $\mathrm{Lie}(\mathrm{GL}(n,\mathbb{R}))$ as well as the matrix group $M(n,\mathbb{R})$.

Example 4 (Lie algebra of the orthogonal group)**.** Consider the Lie subgroup $O(n) \subseteq \mathrm{GL}(n, \mathbb{R})$ consisting of all $n \times n$ orthogonal matrices. Evidently $O(n) = \phi^{-1}(I)$ where $\phi : \mathrm{GL}(n, \mathbb{R}) \to M(n, \mathbb{R})$ is the smooth map given by $\phi(A) = A^T A$, so $T_I O(n) = \ker d\phi_I$. By Fact 11 we can identify $\mathrm{Lie}(O(n))$ with the Lie subalgebra

$$\begin{aligned}\mathrm{Lie}(O(n)) &\simeq \{X \in \mathfrak{gl}(n, \mathbb{R}) : X_e \in T_I O(n)\} \subseteq \mathfrak{gl}(n, \mathbb{R}) \\ &= \{X \in \mathfrak{gl}(n, \mathbb{R}) : X_e \in \ker d\phi_I\}\end{aligned}$$

For any $B \in T_I GL(n, \mathbb{R}) \simeq M(n, \mathbb{R})$ we can calculate $d\phi_I(B)$ using Fact 3 by choosing any smooth curve γ in $M(n, \mathbb{R})$ with $\gamma(0) = I$ and $\gamma'(0) = B$. Evidently $\gamma(t) = I + tB$ works. Thus we calculate

$$(\phi \circ \gamma)(t) = (I + tB)^T (I + tB) = I + t(B + B^T) + t^2 B^T B,$$

from which we can immediately identify the derivative as the linear term $B + B^T$. Therefore

$$d\phi_I(B) = (\phi \circ \gamma)'(0) = B + B^T$$

for any $B \in M(n, \mathbb{R})$, and $B \in \ker d\phi_I$ if and only if $B^T = -B$; that is, if and only if B is skew-symmetric. Hence $\mathrm{Lie}(O(n))$ is isomorphic to the Lie subalgebra $\mathfrak{o}(n) \subseteq \mathfrak{gl}(n, \mathbb{R})$ consisting of all $n \times n$ skew-symmetric matrices.

Fact 14 (Lie algebra of the complex general linear group)**.** *The composition of natural isomorphisms*

$$\mathrm{Lie}(\mathrm{GL}(n, \mathbb{C})) \simeq T_I \, \mathrm{GL}(n, \mathbb{C}) \simeq M(n, \mathbb{C})$$

yields a Lie algebra isomorphism.

Proof. We can use our knowledge about the Lie algebra of the real general linear group to simplify this proof considerably. ■

5 The exponential map

We start by recalling some basic definitions and facts about integral curves and flows of smooth vector fields. Let M be a smooth manifold and X a smooth vector field on M. For any $p \in M$, an ***integral curve*** for X starting at p is a smooth curve $\gamma : (-\epsilon, \epsilon) \to M$ defined in some open interval around 0 satisfying $\gamma(0) = p$ and $\gamma'(t) = X_{\gamma(t)}$ for every $t \in (-\epsilon, \epsilon)$.

Note that by existence and uniqueness for ODEs, for any smooth vector field X and $p \in M$ we can always find an integral curve for X starting at p in some sufficiently small open interval around 0; i.e. integral curves always exist locally. An integral curve for X starting at p is maximal if it cannot be extended to an integral curve on any larger open interval in $\mathbb{R}$.

A fundamental theorem in differential geometry says that for any smooth vector field X on M and for any $p \in M$, there exists a unique maximal integral curve for X starting at p. We let $\theta^{(p)}$ denote this unique maximal integral curve, and denote its domain by $D^{(p)} \subseteq \mathbb{R}$. Define the ***flow domain*** for X as the subset $D \subseteq \mathbb{R} \times M$ such that

$$D^{(p)} = \{t \in \mathbb{R} : (t, p) \in D\}$$

for every $p \in M$. The ***flow*** of X is the smooth map $\theta : D \to M$ given by

$$\theta(t, p) = \theta^{(p)}(t) = \theta_t(p)$$

obtained by "flowing along" the maximal integral curve for X starting at p for some time t. It's straightforward to check the following properties of the flow:

(a) $\theta_0 = \mathrm{id}_M$.

(b) θ_t is a diffeomorphism (from an open subset of M to another open subset of M).

(c) For all $s \in D^{(p)}$ and $t \in D^{(\theta(s,p))}$ such that $s+t \in D^{(p)}$ we have $\theta(t, \theta(s,p)) = \theta(s+t, p)$.

Technically speaking the thing we've defined here is the *maximal* flow of X on a *maximal* flow domain, and we should distinguish between the many different local flows that could be defined with different flow domains. The maximal flow will be sufficient for our purposes though, so we will stick with this slightly simpler picture.

The situation becomes especially simple if all of the integral curves of X are defined on all of $\mathbb{R}$ (which holds, for example, if M is a compact manifold). In this case the flow $\theta : \mathbb{R} \times M \to M$ of X is just a smooth left $\mathbb{R}$-action on M (called a global flow), because we have $\theta_s \circ \theta_t = \theta_{s+t}$ for every $s, t \in \mathbb{R}$ and $\theta_0 = \mathrm{id}_M$. In particular, each map $\theta_t : M \to M$ is a diffeomorphism of M. A smooth vector field X on M is ***complete*** if all of its integral curves are defined on all of $\mathbb{R}$; i.e. if X generates a global flow.

We should make note of one more detail. Say we have an a smooth vector field X on M, and an integral curve $\gamma : (-\epsilon, \epsilon) \to M$ starting at $p \in M$. A priori, the number $\epsilon > 0$ depends on p, and an integral curve starting at some other $q \neq p$ may be defined on a smaller interval. On the other hand, if there exists an $\epsilon > 0$ such that every integral curve of X is defined on (at least) the interval $(-\epsilon, \epsilon)$, then X is actually complete. This is the content of the following lemma:

Lemma 1 (Uniform time lemma)**.** *Let X be a smooth vector field on a smooth manifold M, and let $\theta : D \times M \to M$ denote the flow of X. Suppose there exists an $\epsilon > 0$ such that the domain of $\theta^{(p)}$ contains $(-\epsilon, \epsilon)$ for every $p \in M$. Then X is complete. In particular, θ is a global flow.*

Proof. Suppose for the sake of contradiction that for some $p \in M$, the flow domain $D^{(p)} \subseteq \mathbb{R}$ of $\theta^{(p)}$ is bounded above, and let $b = \sup D^{(p)} < \infty$. By assumption, $\theta^{(p)}(t)$ is defined at least for $t \in (-\epsilon, \epsilon)$. Choose any $t_0 \in (b - \epsilon, b)$ and set $q = \theta^{(p)}(t_0)$. Define a curve $\gamma : (-\epsilon, t_0 + \epsilon) \to M$ by

$$\gamma(t) = \begin{cases} \theta^{(p)}(t) & \text{if } -\epsilon < t < b \\ \theta^{(q)}(t - t_0) & \text{if } t_0 - \epsilon < t < t_0 + \epsilon \end{cases}$$

noting that $\theta^{(q)}$ is defined at each $t - t_0 \in (-\epsilon, \epsilon)$ by assumption. The two definitions agree on their overlap: if $t_0 - \epsilon < t < b$ then

$$\begin{aligned} \theta^{(q)}(t - t_0) &= \theta_{t-t_0}(q) \\ &= \theta_{t-t_0}(\theta^{(p)}(t_0)) \\ &= \theta_{t-t_0}(\theta_t(p)) \\ &= \theta^{(p)}(t) \end{aligned}$$

Evidently γ is an integral curve for X starting at p, and since $t_0 + \epsilon > b$ we have contradicted the assumption that the maximal flow domain $D^{(p)}$ was bounded above by b. A similar argument shows that $D^{(p)}$ cannot be bounded below, so every integral curve is defined on all of $\mathbb{R}$ and X is complete. ■

Our main use for the uniform time lemma is the following fact: the Lie algebra of a Lie group consists of complete vector fields. An essential aspect of the proof of this fact is that for any $g \in G$ and $X \in \mathrm{Lie}(G)$, the left multiplication L_g maps integral curves of X to integral curves of X. Indeed, since X is left-invariant we have

$$d(L_g)_{g'}(X_{g'}) = X_{gg'}$$

for every $g, g' \in G$, so if α is an integral curve for X then $\alpha'(t) = X_{\alpha(t)}$ for every t and therefore

$$\begin{aligned}\frac{d}{dt}(L_g(\alpha(t))) &= d(L_g)_{\alpha(t)}(\alpha'(t)) \\ &= d(L_g)_{\alpha(t)}(X_{\alpha(t)}) \\ &= X_{g\alpha(t)}\end{aligned}$$

so $(g\alpha)'(t) = X_{g\alpha(t)}$ which means that $L_g(\alpha(t)) = g\alpha(t)$ is an integral curve of X (defined on the same interval as α). This is basically a reflection of the *homogeneity* of the Lie group G; more generally one can use the same arguments to prove that a left-invariant vector field on a homogeneous space is complete.

Fact 15. *Let G be a Lie group. Every left-invariant vector field $X \in \mathrm{Lie}(G)$ is complete.*

Proof. Let $\theta : D \to G$ denote the flow of X. There is some $\epsilon > 0$ such that $\theta^{(e)}$ is defined on $(-\epsilon, \epsilon)$. Let $g \in G$ be arbitrary. Since X is left-invariant, L_g maps integral curves of X to integral curves of X, hence $L_g \circ \theta^{(e)}$ is an integral curve of X starting at g, still defined on the interval $(-\epsilon, \epsilon)$. Thus $L_g \circ \theta^{(e)} = \theta^{(g)}$ by uniqueness of integral curves. We conclude that for every $g \in G$, the integral curve $\theta^{(g)}$ is defined on $(-\epsilon, \epsilon)$, so X is complete by Lemma 1. ■

Let G be any Lie group. A ***one-parameter subgroup*** for G is a Lie group homomorphism $\gamma : \mathbb{R} \to G$; i.e. a smooth curve in G which respects the group operation.

Fact 16. *Let G be a Lie group. The one-parameter subgroups for G are precisely the maximal integral curves of left-invariant vector fields on G starting at the identity $e \in G$.*

Proof. Let $X \in \mathrm{Lie}(G)$ and let γ denote the maximal integral curve of X starting at $\gamma(0) = e$. Since left-invariant vector fields are complete, γ is defined on all of $\mathbb{R}$. We just need to show that γ is a group homomorphism. Since X is left-invariant, each left-multiplication diffeomorphism $L_g : G \to G$ maps integral curves of X to integral curves of X, and applying this with $g = \gamma(s)$ for some $s \in \mathbb{R}$ we see that $t \mapsto L_{\gamma(s)}(\gamma(t)) = \gamma(s)\gamma(t)$ is an integral curve of X starting at $\gamma(s)$. But clearly $t \mapsto \gamma(s + t)$ is also an integral curve of X starting at $\gamma(s)$, so we conclude that $\gamma(s + t) = \gamma(s)\gamma(t)$ for every $s, t \in \mathbb{R}$ by the uniqueness of integral curves. Thus γ is a one-parameter subgroup for G.

Conversely, suppose $\gamma : \mathbb{R} \to G$ is a one-parameter subgroup for G, and let $\gamma_* : \mathrm{Lie}(\mathbb{R}) \to \mathrm{Lie}(G)$ denote the induced Lie algebra homomorphism. Define a left-invariant vector field on G by

$$X = \gamma_* \left(\frac{d}{dt} \right)$$

so that, by definition of γ_* we have

$$X_{\gamma(t_0)} = \gamma_* \left(\frac{d}{dt} \right) \bigg|_{\gamma(t_0)} = d\gamma_{t_0} \left(\frac{d}{dt}\bigg|_{t=t_0} \right) = \gamma'(t_0)$$

for every $t_0 \in \mathbb{R}$. Thus γ is an integral curve for X starting at $\gamma(0) = e$. ■

Given $X \in \mathrm{Lie}(G)$ we call the maximal integral curve of X starting at $e \in G$ the ***one-parameter subgroup for G generated by X***. Since a left-invariant vector field is determined by its value at $e \in G$, it follows that a one-parameter subgroup for G is determined by its initial velocity in T_eG. We have established a three-way correspondence:

$$\{\text{one-parameter subgroups for } G\} \longleftrightarrow \mathrm{Lie}(G) \longleftrightarrow T_eG$$

We can explicitly compute the one-parameter subgroups for $\mathrm{GL}(n, \mathbb{R})$: they are generated by matrix exponentials. For any matrix $A \in M(n, \mathbb{R})$, we define

$$e^A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k = I + A + \frac{1}{2} A^2 + \cdots$$

called the ***exponential of*** A. A priori, this is merely a formal sum, but in fact the series converges to an invertible matrix $e^A \in \mathrm{GL}(n, \mathbb{R})$ (where convergence is taken with respect to the Frobenius norm on $M(n, \mathbb{R})$). To see why, note that the Frobenius norm on $M(n, \mathbb{R})$ satisfies $|AB| \leq |A||B|$, hence $|A^k| \leq |A|^k$ for every $k \geq 1$. Moreover, we have a convergent series

$$\sum_{k=0}^{\infty} \frac{1}{k!} |A|^k = e^{|A|} < \infty$$

Therefore by the Weierstrass M-test the series $\sum A^k/k!$ converges uniformly to a well-defined matrix e^A for any $A \in M(n, \mathbb{R})$. The matrix exponential e^A is clearly invertible because its eigenvalues are $\{e^{\lambda_i}\}$ where $\{\lambda_i\}$ are the eigenvalues of A (so, none of them are zero). In fact, we will show below that $e^{-A} = (e^A)^{-1}$.

Fact 17 (One-parameter subgroups of $\mathrm{GL}(n, \mathbb{R})$)**.** *For any $A \in \mathfrak{gl}(n, \mathbb{R})$, the one-parameter subgroup for $\mathrm{GL}(n, \mathbb{R})$ generated by A is the function $\gamma(t) = e^{tA}$.*

Proof. Identifying $A \in \mathfrak{gl}(n, \mathbb{R})$ with a matrix $A = (A_{ij}) \in M(n, \mathbb{R})$, we get a corresponding left-invariant vector field A^L on $\mathrm{GL}(n, \mathbb{R})$ given by

$$A^L(B) = d(L_B)_I(A) = BA$$

By Fact 16, the one-parameter subgroup for $\mathrm{GL}(n, \mathbb{R})$ generated by A is the integral curve of A^L starting at I. In other words, it is the (unique) solution of the initial value problem:

$$\begin{cases} \gamma'(t) = A^L(\gamma(t)) = \gamma(t)A \\ \gamma(0) = I \end{cases}$$

We just need to check two things: that $\gamma(t) = e^{tA}$ is invertible for every $t \in \mathbb{R}$ so that it defines a function $\mathbb{R} \to \mathrm{GL}(n, \mathbb{R})$, and that γ satisfies this differential equation – i.e. that $\frac{d}{dt}(e^{tA}) = e^{tA}A$ for every $A \in M(n, \mathbb{R})$.

The series $e^{tA} = \sum_{k=0}^{\infty}(t^k/k!)A^k$ is a (uniformly) convergent series of functions $\mathbb{R} \to M(n, \mathbb{R})$, and the series of derivatives is

$$\begin{aligned}
\sum_{k=0}^{\infty} \frac{1}{k!} \frac{d}{dt}(t^k A^k) &= \sum_{k=0}^{\infty} \frac{1}{k!} k t^{k-1} A^k \\
&= \sum_{k=1}^{\infty} \frac{1}{(k-1)!} t^{k-1} A^k \\
&= \sum_{k=0}^{\infty} \frac{1}{k!} t^k A^{k+1} \\
&= A \sum_{k=0}^{\infty} \frac{1}{k!} t^k A^k
\end{aligned}$$

which obviously converges absolutely and uniformly to Ae^{tA}. Thus, term-by-term differentiation is justified and we conclude that

$$\begin{aligned}
\frac{d}{dt}(e^{tA}) &= \frac{d}{dt}\left(\sum_{k=0}^{\infty} \frac{1}{k!} t^k A^k\right) \\
&= \sum_{k=0}^{\infty} \frac{1}{k!} \frac{d}{dt}(t^k A^k) \\
&= Ae^{tA}
\end{aligned}$$

Note that this calculation also shows that $Ae^{tA} = e^{tA}A$ because in the preceding calculation we could factor out A on either side of the summation and obtain the same end result. By smoothness of solutions to ODEs, we also find that $\gamma(t) = e^{tA}$ is a smooth map $\mathbb{R} \to M(n, \mathbb{R})$.

Finally, we show that e^{tA} is invertible for every $t \in \mathbb{R}$, so that γ defines a smooth curve in $\mathrm{GL}(n, \mathbb{R})$. Let $\sigma(t) = \gamma(t)\gamma(-t)$. For any $t \in \mathbb{R}$ we have

$$\begin{aligned}
\sigma'(t) &= \gamma'(t)\gamma(-t) - \gamma(t)\gamma'(-t) \\
&= Ae^{tA}e^{-tA} - e^{tA}Ae^{-tA} \\
&= Ae^{tA}e^{-tA} - Ae^{tA}e^{-tA} \\
&= 0
\end{aligned}$$

since A commutes with e^{tA}. Thus $\sigma(t)$ is constant, and clearly $\sigma(0) = I$, so $\sigma(t) = I$ for every $t \in \mathbb{R}$. We conclude that $(e^{tA})^{-1} = e^{-tA}$ and therefore $\gamma(t) = e^{tA}$ is the one-parameter subgroup for $\mathrm{GL}(n, \mathbb{R})$ generated by A. ■

Motivated by the matrix exponential which characterizes the one parameter subgroups for $\mathrm{GL}(n, \mathbb{R})$, we will define an exponential map for any Lie group G so that the exponentials of left-invariant vector fields on G generate its one-parameter subgroups. Given a Lie group G with Lie algebra $\mathfrak{g}$, the ***exponential map*** for G is the function $\exp : \mathfrak{g} \to G$ defined by

$$\exp(X) = \gamma(1)$$

where γ is the one-parameter subgroup for G generated by X (i.e. the integral curve of X starting at $e \in G$). It follows immediately from this definition that for any $X \in \mathfrak{g}$, the curve $\gamma(s) = \exp(sX)$ is the one-parameter subgroup for G generated by X. Thus, Fact 17 shows that the exponential map for $\mathrm{GL}(n, \mathbb{R})$ is exactly the matrix exponential $\exp(A) = e^A$.

Fact 18. *Let G be a Lie group and $H \subseteq G$ a Lie subgroup. The one-parameter subgroups for H are precisely those one-parameter subgroups for G whose initial velocities lie in T_eH.*

Proof. If $\gamma : \mathbb{R} \to H$ is a one-parameter subgroup for H then it is obviously a group homomorphism mapping into G, so it's a one-parameter subgroup for G satsifying $\gamma'(0) \in T_eH$. Conversely, let $\gamma : \mathbb{R} \to G$ be a one-parameter subgroup for G with $\gamma'(0) \in T_eH$. ■

Fact 19 (Properties of the exponential map)**.** *Let G be a Lie group with Lie algebra $\mathfrak{g}$. Then*

(i) $\exp : \mathfrak{g} \to G$ *is a smooth map.*

(ii) For any $X \in \mathfrak{g}$ and $s, t \in \mathbb{R}$ we have $\exp((s+t)X) = \exp(sX)\exp(tX)$.

(iii) For any $X \in \mathfrak{g}$, we have $(\exp X)^{-1} = \exp(-X)$.

(iv) More generally, for any $X \in \mathfrak{g}$ and $n \in \mathbb{Z}$, we have $(\exp X)^n = \exp(nX)$.

(v) The differential $d(\exp)_0 : T_0\mathfrak{g} \to T_eG$ is the identity map (identifying both $T_0\mathfrak{g}$ and T_eG with $\mathfrak{g}$ itself).

(vi) $\exp$ *restricts to a diffeomorphism from some neighborhood of* 0 *in* $\mathfrak{g}$ *to a neighborhood of $e \in G$.*

Proof. (i) Define a map $\phi : \mathbb{R} \times (G \times \mathfrak{g}) \to G \times \mathfrak{g}$ by

$$\phi(t, g, X) = (g \cdot \exp(tX), X),$$

and note that this is the flow of the left-invariant vector field $(X, 0)$ on $G \times \mathfrak{g}$. Thus it is smooth as the flow of a smooth vector field. Now we can decompose exp as

$$\exp = \pi_1 \circ \phi \circ i = \mathfrak{g} \overset{i}{\hookrightarrow} \mathbb{R} \times G \times \mathfrak{g} \xrightarrow{\phi} G \times \mathfrak{g} \xrightarrow{\pi_1} G$$

because

$$\pi_1(\phi(i(X))) = \pi_1(\phi(1, e, X)) = \pi_1(\exp(X), X) = \exp(X)$$

for every $X \in \mathfrak{g}$. We conclude that exp is smooth as a composition of smooth maps.

(ii) Since $t \mapsto \exp(tX)$ is by definition the one-parameter subgroup for G generated by X, this map is a group homomorphism $\mathbb{R} \to G$, and property (ii) follows immediately.

(iii) This property follows immediately from property (ii). One can calculate directly that $e = \exp((1-1)X = \exp(X)\exp(-X) = \exp(-X)\exp(X)$.

(iv) For $n = 0$ the statement is self-evident, for positive integers n this follows by induction on (ii), and for negative integers n this follows by induction on (ii) together with (iii).

(v) Let $X \in \mathfrak{g}$ and let $\sigma : \mathbb{R} \to \mathfrak{g}$ be the curve $\sigma(t) = tX$. Then $\sigma'(0) = X$ and we have

$$\begin{aligned}(d\exp)_0(X) &= d(\exp)_0(\sigma'(0)) \\ &= \frac{d}{dt}\bigg|_{t=0} (\exp \circ \sigma) \\ &= \frac{d}{dt}\bigg|_{t=0} \exp(tX) \\ &= X_e\end{aligned}$$

since $t \mapsto \exp(tX)$ is the integral curve for X in G starting at $e \in G$. Recalling that we identify X_e with X under the isomorphism $T_eG \simeq \mathfrak{g}$, we conclude that $d\exp_0 = \mathrm{id}$.

(vi) Since $d\exp_0 = \mathrm{id}$ is invertible by (v), the inverse function theorem says that exp is a local diffeomorphism at $0 \in \mathfrak{g}$, which is another way of saying that exp restricts to a diffeomorphism on a neighborhood of $0 \in \mathfrak{g}$.

■

6 The closed subgroup theorem

In this section we will use the machinery of exponential maps to prove the closed subgroup theorem: given any Lie group G, a subgroup $H \subseteq G$ is an embedded Lie subgroup if and only if it's a topologically closed subset. One direction can be proven using some simple point-set topology: if H is an embedded Lie subgroup then it must be a closed subset. The converse is much more difficult, but by Fact 4 it suffices to show that H is an embedded submanifold whenever H is closed.

For this latter statement: by the local slice criterion (Theorem 1) it suffices to construct a slice chart around any point in H. We will deploy the following lemma to construct a slice chart around the identity $e \in H$, and then we will translate this chart via left multiplication to get a slice chart around any arbitrary point.

Lemma 2. *Let G be a Lie group with Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ and let $H \subseteq G$ be a closed subgroup. Define a Lie subalgebra:*

$$\mathfrak{h} = \{X \in \mathfrak{g} : \exp(tX) \in H \text{ for every } t \in \mathbb{R}\} \subseteq \mathfrak{g}$$

There is an open neighborhood $0 \in U \subseteq \mathfrak{g}$ so that $\exp : U \to \exp(U)$ is a diffeomorphism and $\exp(U \cap \mathfrak{h}) = \exp(U) \cap H$.

Proof. For any open neighborhood $0 \in U \subseteq \mathfrak{g}$ we automatically have $\exp(U \cap \mathfrak{h}) \subseteq \exp(U) \cap \mathfrak{h}$ by definition of $\mathfrak{h}$, so it suffices to show that U can be chosen such that the reverse inclusion also holds. For the sake of contradiction, suppose that the negation holds:

> For every open neighborhood $0 \in U \subseteq \mathfrak{g}$ on which exp is a diffeomorphism onto its image, we have $\exp(U) \cap H \not\subseteq \exp(U \cap \mathfrak{h})$; i.e. there exists an $h \in \exp(U) \cap H$ such that $h \notin \exp(U \cap \mathfrak{h})$. (2)

Choose an inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$ and let $\mathfrak{s}$ be the orthogonal complement of $\mathfrak{h}$ in $\mathfrak{g}$ with respect to this inner product, so that we have a decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{s}$.

Define $\phi : \mathfrak{h} \oplus \mathfrak{s} \to G$ by $\phi(X, Y) = \exp(X)\exp(Y)$, which is a diffeomorphism near $(0, 0)$. Choose open neighborhoods $0 \in U_0 \subseteq \mathfrak{g}$ and $(0, 0) \in W_0 \subseteq \mathfrak{h} \oplus \mathfrak{s}$ so that $\exp|_{U_0}$ and $\phi|_{W_0}$ are both diffeomorphisms onto their images. Let $\{U_i\}$ be a countable neighborhood basis for $\mathfrak{g}$ at 0 (for instance, a sequence of coordinate balls whose radii approach zero), and set

$$V_i = \exp(U_i) \subseteq G$$
$$W_i = \phi^{-1}(V_i) \subseteq \mathfrak{h} \oplus \mathfrak{s}$$

so that $\{V_i\}$ is a countable neighborhood basis for $\mathfrak{h} \oplus \mathfrak{s}$ at $(0, 0)$, and $\{U_i\}$ is a countable neighborhood basis for G at e. By shrinking the sets U_i if necessary, we can assume without loss of generality that $U_i \subseteq U_0$ and $W_i \subseteq W_0$ for every i.

By our assumption (2), for each i we can find an $h_i \in \exp(U_i) \cap H$ such that $h_i \notin \exp(U_i \cap \mathfrak{h})$. Write $h_i = \exp(z_i) \in H$ for some $z_i \in U_i$. Since $\exp(U_i) = V_i = \phi(W_i)$ we can also write

$$h_i = \exp(z_i) = \phi(X_i, Y_i) = \exp(X_i)\exp(Y_i)$$

for some $(X_i, Y_i) \in W_i$. Notice that $Y_i \neq 0$, otherwise we would have $\exp(X_i) = h_i = \exp(z_i)$ hence $X_i = z_i \in U_i \cap \mathfrak{h}$ which implies that $h_i \in \exp(U_i \cap \mathfrak{h})$, contradicting (2).

Let $|\cdot|$ denote the norm on $\mathfrak{g}$ associated with the chosen inner product. Define $c_i = |Y_i|$ and note that $c_i \to 0$ since $Y_i \to 0$. Then the sequence $(c_i^{-1}Y_i)$ lies on the unit sphere in $\mathfrak{s}$, and by compactness we can assume without loss of generality that $c_i^{-1}Y_i$ converges to some $Y \in \mathfrak{s}$ with $|Y| = 1$. Let's verify that $\exp(tY) \in H$ for every $t \in \mathbb{R}$ – the desired contradiction will follow from this, because then by definition of $\mathfrak{h}$ we will have $Y \in \mathfrak{h} \cap \mathfrak{s} = \{0\}$ which is impossible since $|Y| = 1$.

Fix any $t \in \mathbb{R}$ and let $n_i = \lfloor t/c_i \rfloor$ for each i. Then

$$|n_i - t_i/c_i| \leq 1 \;\Rightarrow\; |n_i c_i - t_i| \leq c_i \to 0$$

so $n_i c_i \to t$ as $i \to \infty$. Therefore

$$n_i Y_i = (n_i c_i)(c_i^{-1} Y_i) \to tY$$

which implies that $\exp(Y_i)^{n_i} = \exp(n_i Y_i) \to \exp(tY)$ as $i \to \infty$. But $\exp(Y_i) = \exp(X_i)^{-1} h_i \in H$ for every i, and H is a closed subgroup, so we conclude that $\exp(tY) \in H$ too. This obtains a contradiction and the proof is complete. ∎

Of course, the subalgebra $\mathfrak{h}$ we defined in Lemma 2 turns out to be the Lie algebra of H, but we haven't yet proven that H is actually a Lie group. This next lemma will be used to establish the relatively easy direction "an embedded Lie subgroup is toplogically closed".

Lemma 3. *Let G be a Lie group, and let $W \subseteq G$ be any open neighborhood of the identity element $e \in G$. There exists a smaller open neighborhood $V \subseteq W$ around e with the property that for any $g, h \in V$ we have $gh^{-1} \in W$.*

Proof. Define $\varphi : G \times G \to G$ by $\varphi(g, h) = gh^{-1}$. Since φ is continuous and $W \subseteq G$ is open, the pre-image $\varphi^{-1}(W) \subseteq G \times G$ is open in the product space. Write $\varphi^{-1}(W) = W_1 \times W_2$ where $W_1, W_2 \subseteq G$ are open neighborhoods of the identity in G (they contain e since $\varphi(e, e) = e \in W$ implies that $(e, e) \in \varphi^{-1}(W)$). Choose $V = W_1 \cap W_2 \cap W$ – this is obviously an open neighborhood of e contained inside W, and since $V \times V \subseteq W_1 \times W_2 = \varphi^{-1}(W)$ we have $\varphi(V \times V) \subseteq W$. If $g, h \in V$ then $(g, h) \in V \times V$ implies that $gh^{-1} = \varphi(g, h) \in W$. ∎

Theorem 2 (Closed subgroup theorem)**.** *Let G be a Lie group and $H \subseteq G$ any subgroup. Then H is an embedded Lie subgroup of G if and only if H is closed.*

Proof. First suppose that H is an embedded Lie subgroup of G. We want to show that H is a closed subset. Let (h_i) be any sequence in H converging to some $g \in \overline{H}$. We want to show that $g \in H$. Since H is embedded, by the local slice criterion (Theorem 1) we can choose a slice chart $U \subseteq G$ for H around $e \in H$, and then choose a smaller open neighborhood W around e such that $\overline{W} \subseteq U$. By Lemma 3 we can find an open neighborhood $V \subseteq W$ around e such that $g_1 g_2^{-1} \in W$ whenever $g_1, g_2 \in V$.

Now $h_i \to g$ implies that $h_i g^{-1} \to e$ by continuity of the multiplication in G, so the sequence $(h_i g^{-1}$ eventually lies in V, and we assume without loss of generality that $(h_i g^{-1}) \subseteq V$. Therefore

$$h_i h_j^{-1} = h_i g^{-1} g h_j^{-1} = h_i g^{-1} (h_j g^{-1})^{-1} \in W \cap H$$

for every i and j. Fixing j and letting $i \to \infty$, we find that

$$h_i h_j^{-1} \to g h_j^{-1} \in \overline{W} \cap \overline{H} \subseteq U \cap \overline{H}.$$

Since U is a slice chart for H in G, the slice $U \cap H$ is diffeomorphic to an affine subspace of (an open subset of) Euclidean space, hence $U \cap H$ is closed in U, and therefore $U \cap \overline{H} = U \cap H$. As a result, we have $g h_j^{-1} \in H$ implying $g \in H$, thus $\overline{H} = H$ and H is closed.

Conversely, suppose that H is a closed subgroup of G. By Fact 4 (embedded subgroups are Lie subgroups), it suffices to show that H is an embedded submanifold, hence by the local slice criterion it suffices to construct a slice chart for H around any point in H. First we construct a slice chart around the identity $e \in H$. Let $\mathfrak{h} \subseteq \mathfrak{g}$ be the Lie subalgebra defined in Lemma 2, and let U be the open neighborhood of 0 in $\mathfrak{g}$ given by the lemma. Choose any linear isomorphism $E : \mathfrak{g} \to \mathbb{R}^n$ mapping $\mathfrak{h} \subseteq \mathfrak{g}$ to $\mathbb{R}^k \subseteq \mathbb{R}^n$. Then $\varphi = E \circ \exp^{-1} : \exp(U) \to \mathbb{R}^n$ is a smooth chart for G around e, and

$$\begin{aligned} \varphi(\exp(U) \cap H) &= E(\exp^{-1}(\exp(U) \cap H)) \\ &= E(U \cap \mathfrak{h}) \end{aligned}$$

where the latter set is diffeomorphic to an open subset of the affine space in $\mathbb{R}^n$ consisting of those points whose last $n-k$ coordinates are zero (by definition of the map E). Thus $(\varphi, \exp(U) \cap H)$ is a slice chart for H around e.

Now for any arbitrary point $h \in H$, the left translation diffeomorphism $L_h : H \to H$ maps $\exp(U) \cap H$ onto $L_h(\exp U) \cap H$. Thus $(\varphi \circ L_h^{-1}, L_h(\exp U) \cap H)$ is a slice chart for H around h. ■

Theorem 2 was first published by Élie Cartan (the father) in 1930, hence the theorem is often called Cartan's theorem or Cartan's closed subgroup theorem.

Corollary 1. *Let G be a Lie group. For any subgroup $H \subseteq G$, the closure $\overline{H}$ is an embedded Lie subgroup of G.*

As an immediate consequence, we deduce that all Lie subgroups are either embedded, or they are dense subgroups of embedded Lie subgroups. The prototypical example: the image of an irrational line on the torus under the embedding $\mathbb{T}^2 \hookrightarrow \mathbb{T}^3$. It's a Lie subgroup of $\mathbb{T}^3$ which is not closed, embedded, or dense, but it's obviously dense in its closure, which is an embedded Lie subgroup of $\mathbb{T}^3$.

Summary. Here is a summary of the logic and ultimate results of this section:

1. The relatively easy direction "embedded Lie subgroup is topologically closed" was proved using some point-set topology; namely Theorem 1 and Lemma 3.
2. The nontrivial direction "closed subgroup is an embedded Lie subgroup" utilized our results about Lie algebras and exponential maps. Since embedded subgroups are Lie subgroups (Fact 4), we only needed to show that a closed subgroup is an embedded submanifold. We did this using Theorem 1 and Lemma 2.
3. As a result, we have characterized the embedded Lie subgroups: they are exactly the closed subgroups. We have also characterized the Lie subgroups: they are dense subsets of embedded Lie subgroups.

7 References

In this note we mostly followed John Lee's *Intro to Smooth Manifolds* (pp. 181-199 and pp. 515-525), filling in details to several exercises and problems along the way.