Lie groups and quotient manifolds

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Abstract

We present all of the necessary tools and techniques from the theory of smooth manifolds and Lie groups to state the quotient manifold theorem: given a smooth, proper, free Lie group action on a smooth manifold, the orbit space is a smooth manifold. Then we explore some important applications of the theorem, such as the construction and characterization of homogeneous spaces.

Contents

1 Preliminaries

Recall that a smooth map $F : M \to N$ is a ***submersion*** at $x \in M$ if the differential $dF_x : T_xM \to T_{F(x)}N$ is surjective. If F is a submersion at every point on M, we simply call it a smooth submersion. We have the following local characterization for submersions:

Theorem 1 (Local submersion theorem)**.** *Let M and N be smooth manifolds of dimension m and n respectively. Suppose $F : M \to N$ is a smooth submersion* *at $x \in M$, and let $y = F(x)$. Then there exist smooth charts φ around x and ψ around y such that the local coordinate representation $\widehat{F} = \psi \circ F \circ \varphi^{-1}$ is*

$$\widehat{F}(x_1, \ldots, x_n, x_{n+1}, \ldots, x_m) = (x_1, \ldots, x_n).$$

In other words, F is locally equivalent to the projection $\mathbb{R}^m \to \mathbb{R}^n$ near x.

Suppose M is a smooth manifold. An ***embedded submanifold*** of M is a subset $S \subseteq M$ that is a topological manifold with respect to the subspace topology, and equipped with a smooth structured such that the inclusion $S \hookrightarrow M$ is a smooth embedding. Note that embedded submanifolds $S \subseteq M$ are precisely the images $S = F(N)$ of smooth embeddings $F : N \to M$.

An ***immersed submanifold*** of M is a subset $S \subseteq M$ that is a topological manifold (not necessarily with them subspace topology), equipped with a smooth structure such that the inclusion $S \hookrightarrow M$ is a smooth immersion. Analogous to the embedded case, we note that immersed submanifolds $S \subseteq M$ are precisely the images $S = F(N)$ of injective smooth immersions $F : N \to M$.

Recall that the ***rank*** of a smooth map $F : M \to N$ at $p \in M$ is the dimension of the image of dF_p inside $T_{F(p)}N$.

Theorem 2 (Constant-rank level set theorem)**.** *Let M and N be smooth manifolds, and let $F : M \to N$ be a smooth map of constant rank r. Then each level set $F^{-1}(y)$ is an embedded submanifold of codimension r in M.*

Theorem 3 (Global rank theorem)**.** *Let M and N be smooth manifolds, and let $F : M \to N$ be a smooth map of constant rank. Then:*

(i) If F is surjective, then it is a smooth submersion.

(ii) If F is injective, then it is a smooth immersion.

(iii) If F is bijective, then it is a diffeomorphism.

Let $U \subseteq \mathbb{R}^n$ be open. A ***k-slice*** of U is a subset of the form

$$S = \{(x_1, \ldots, x_k, c_{k+1}, \ldots, c_n) \in U\} \subseteq U$$

for some constants $c_{k+1}, \ldots, c_n \in \mathbb{R}$. Thus, a k-slice is an affine subset of U homeomorphic to an open subset of $\mathbb{R}^k$. Now take a smooth n-manifold M and let $\varphi : U \subseteq M \to \mathbb{R}^n$ be a smooth chart on M. A ***k-slice*** of $U \subseteq M$ is a subset $S \subseteq U$ such that $\varphi(S)$ is a k-slice of $\varphi(U) \subseteq \mathbb{R}^n$.

We say that a subset $N \subseteq M$ satisfies the ***local k-slice condition*** if: for every $p \in N$ there exists a smooth chart $\varphi : U \subseteq M \to \mathbb{R}^n$ for M around p such that $N \cap U$ is a k-slice of U; i.e. $\varphi(N \cap U)$ is a k-slice of $\varphi(U)$. Such a chart (U, φ) is called a ***slice chart*** for N in M. Roughly speaking, the local k-slice condition says that, near any point, N is locally homeomorphic to a k-dimensional affine open subset of $\mathbb{R}^n$.

Theorem 4 (Local slice criterion)**.** *Let M be a smooth manifold. A subset $N \subseteq M$ is an embedded k-dimensional submanifold of M if and only if N satisfies the local k-slice condition.*

Recall that the differential of a smooth map can be calculated using smooth curves:

Fact 1. *Let $F : M \to N$ be a smooth map and let $p \in M$. For any $v \in T_pM$ we have*

$$dF_p(v) = (F \circ \gamma)'(0)$$

for any smooth curve $\gamma : I \to M$ satisfying $\gamma(0) = p$ and $\gamma'(0) = v$.

2 Lie groups

A ***Lie group*** is a set G which is both a smooth manifold and a group, such that the multiplication and inversion maps

$$m : G \times G \to G, \ \ m(g,h) = gh$$
$$j : G \to G, \ \ j(g) = g^{-1}$$

are both smooth. In this case j is a diffeomorphism because it is its own inverse. In general we will denote the identity element of a group by $e \in G$, but often times there is a more common notation available like 0 for an abelian group, I_n for the $n \times n$ identity matrix of a matrix group, or 1 for the multiplicative group of real or complex numbers.

Fact 2 (Differential of multiplication and inversion)**.** *Let G be a Lie group. Then the differentials $dm_{(e,e)} : T_eG \times T_eG \to T_eG$ and $dj_e : T_eG \to T_eG$ of the multiplication and inversion maps are given by*

$$dm_{(e,e)}(X,Y) = X + Y$$
$$dj_e(X) = -X$$

for every $X, Y \in T_eG$.

Proof. Let $i_1 : G \simeq G \times \{e\} \hookrightarrow G \times G$ and $i_2 : G \simeq \{e\} \times \{G\} \hookrightarrow G \times G$ denote the natural inclusions. Under the identification $T_{(e,e)}(G \times G) \simeq T_eG \times T_e(G)$ we have for any $f \in C^\infty(G)$

$$\begin{aligned} dm_{(e,e)}(X,0)f &= (X,0) \cdot (f \circ m) \\ &= X \cdot (f \circ m \circ i_1) + 0 \cdot (f \circ m \circ i_2) \\ &= X(f) \end{aligned}$$

because $m \circ i_1 = m \circ i_2 = \mathrm{id}_G$. The same calculation shows that $dm_{(e,e)}(0,Y)f = Yf$ and therefore

$$dm_{(e,e)}(X,Y)f = [dm_{(e,e)}(X,0) + dm_{(e,e)}(0,Y)]f = Xf + Yf$$

which shows that $dm_{(e,e)}(X,Y) = X + Y$. A similar argument works for the inversion map. ■

A crucial fact about Lie groups is that any point can be mapped to any other via a diffeomorphism (i.e. Lie groups are homogeneous spaces). Indeed, take a Lie group G and for any $g \in G$ define the maps

$$L_g : G \to G, \ \ L_g(h) = gh$$
$$R_g : G \to G, \ \ R_g(h) = hg$$

called ***left translation*** and ***right translation*** by g, respectively. Note that L_g is smooth because it can be expressed as a composition of an inclusion with multiplication,

$$L_g = m \circ i : G \simeq \{g\} \times G \overset{i}{\hookrightarrow} G \times G \xrightarrow{m} G$$

Moreover, L_g is a diffeomorphism because its inverse is $L_g^{-1} = L_{g^{-1}}$ which is also smooth. A similar argument shows that the right translation map R_g is a diffeomorphism. For emphasis we summarize the preceding observations in the following fact.

Fact 3. *Let G be a Lie group. For any $g \in G$ the left and right translation maps L_g and R_g are diffeomorphisms.*

Example 1 (Lie groups)**.**

(a) The Euclidean space $\mathbb{R}^n$ is a Lie group with respect to vector addition because the coordinates of $x - y$ are smooth functions of the coordinates of (x, y). For the same reason, $\mathbb{C}^n$ is a Lie group with respect to addition.

(b) The general linear group $\mathrm{GL}(n, \mathbb{R})$ is a Lie group with respect to matrix multiplication because the entries of AB are polynomials in the entries of A and B. Inversion is smooth by Cramer's rule. In particular when $n = 1$ this means that the set of nonzero real numbers $\mathbb{R}^\times$ is a Lie group with respect to multiplication.

(c) The complex general linear group $\mathrm{GL}(n, \mathbb{C})$ is an open submanifold of $M(n, \mathbb{C})$, hence it's a smooth manifold of (real) dimension $2n^2$. Matrix multiplication and inversion are smooth functions of the real and imaginary parts of the matrix entries. In particular when $n = 1$ this means that the set of nonzero complex numbers $\mathbb{C}^\times$ is a Lie group with respect to multiplication.

(d) Let V be any vector space over $\mathbb{R}$ or $\mathbb{C}$. The group $\mathrm{GL}(V)$ of invertible linear maps $V \to V$ is also a Lie group.

(e) Let G be a Lie group and $H \subseteq G$ an open subgroup. Then H inherits both the group structure and smooth manifold structure from G, so H is a Lie group.

(f) The group of $n \times n$ invertible matrices over $\mathbb{R}$ with positive determinant is an open subgroup of $\mathrm{GL}(n, \mathbb{R})$ as the pre-image of the open set $(0, \infty) \subset \mathbb{R}$ under the (continuous) determinant map, hence it's an n^2-dimensional Lie group.

(g) $S^1 \subseteq \mathbb{C}^\times$ is a 1-dimensional smooth manifold and a group with respect to complex multiplication. In local coordinates, multiplication and inversion in this group look like

$$(\theta_1, \theta_2) \mapsto \theta_1 + \theta_2 \quad \text{and} \quad \theta \mapsto -\theta$$

so they are smooth operations. Thus S^1 is a Lie group (called the circle group).

(h) The direct product of any finite collection of Lie groups is a Lie group with respect to component-wise operations. For example, the n-dimensional torus $T^n = S^1 \times \cdots S^1$ is a Lie group.

(i) Any group with the discrete topology is a discrete topological group. If the group is finite or countable, then it's also a 0-dimensional smooth manifold, hence it's a discrete Lie group.

Let G and H be Lie groups. A ***Lie group homomorphism*** is a smooth map $F : G \to H$ that is also a group homomorphism. If F is bijective and F^{-1} is also a Lie group homomorphism (i.e. a smooth map, since it's automatically a homomorphism) then F is both a diffeomorphism of smooth manifolds and an isomorphism of groups; in this case we call it a ***Lie group isomorphism***.

Example 2 (Lie group homomorphisms)**.**

(a) If G is a Lie group and $H \subseteq G$ is any Lie subgroup, the inclusion $i : H \hookrightarrow G$ is a Lie group homomorphism. For instance, the inclusion $i : S^1 \hookrightarrow \mathbb{C}^\times$ is a Lie group homomorphism.

(b) The exponential function $\mathbb{R} \to \mathbb{R}^\times$, $t \mapsto e^t$ is an injective Lie group homomorphism because $e^{(s+t)} = e^s e^t$ for any $s, t \in \mathbb{R}$. The image of the exponential function is the open subgroup $\mathbb{R}^+ \subseteq \mathbb{R}^\times$ consisting of positive real numbers. Hence, mapping onto its image, it provides a Lie group isomorphism $\mathbb{R} \simeq \mathbb{R}^+$ with inverse the natural logarithm function $\mathbb{R}^+ \to \mathbb{R}$, $t \mapsto \ln t$.

(c) The complex exponential function $\mathbb{C} \to \mathbb{C}^\times$ given by $z \mapsto e^z$ is a Lie group homomorphism which is surjective but not injective.

(d) The smooth map $\varepsilon : \mathbb{R} \to S^1$ given by $\epsilon(t) = e^{2\pi i t}$ is a Lie group homomorphism with kernel $\ker \varepsilon = \mathbb{Z}$. Similarly, $\varepsilon^n : \mathbb{R}^n \to \mathbb{T}^n$ given by

$$\varepsilon^n(x_1, \ldots, x_n) = (e^{2\pi i x_1}, \ldots, e^{2\pi i x_n})$$

is a Lie group homomorphism with kernel $\ker \varepsilon^n = \mathbb{Z}^n$. Thus, as we will show later, this induces a Lie group isomorphism $\mathbb{R}^n/\mathbb{Z}^n \simeq \mathbb{T}^n$ (by analogy with the first isomorphism theorem for groups).

(e) The determinant function $\det : \mathrm{GL}(n, \mathbb{R}) \to \mathbb{R}^\times$ is a surjective Lie group homomorphism with kernel $\ker \det = \mathrm{SL}(n, \mathbb{R})$.

(f) Let G be any Lie group. For any $g \in G$, the "conjugation by g" map $C_g : G \to G$ given by $x \mapsto gxg^{-1}$ is a Lie group isomorphism of G onto itself.

Fact 4. *Every Lie group homomorphism has constant rank.*

Proof. Let $F : G \to H$ be a Lie group homomorphism and let $e \in G$ and $\widetilde{e} \in H$ denote the identity elements. Fixing an arbitrary $g_0 \in G$, we will show that the rank of F at g_0 is the same as the rank of F at e. First notice that for any $g \in G$ we have

$$F(L_{g_0}(g)) = F(g_0 g) = F(g_0)F(g) = L_{F(g_0)}(F(g))$$

so $F \circ L_{g_0} = L_{F(g_0)} \circ F$. Now take the derivative of both sides at the identity $e \in G$ to see that

$$dF_{g_0} \circ d(L_{g_0})_e = d(F \circ L_{g_0})_e = d(L_{F(g_0)} \circ F)_e = d(L_{F(g_0)})_{\widetilde{e}} \circ dF_e$$

and $d(L_{g_0})_e$ is an isomorphism because L_{g_0} is a diffeomorphism, so

$$dF_{g_0} = d(L_{F(g_0)})_{\widetilde{e}} \circ dF_e \circ d(L_{g_0})_e^{-1}$$

so dF_{g_0} and dF_e have the same rank. ∎

Now combining Fact 4 with the global rank theorem, we know that a constant rank smooth map is bijective if and only if it's a diffeomorphism, so a Lie group homomorphism is bijective if and only if it's a diffeomorphism. Therefore a Lie group homomorphism is bijective if and only if it's a Lie group isomorphism.

Corollary 1. *A Lie group homomorphism is a Lie group isomorphism if and only if it's bijective.*

3 Lie subgroups

Let G be a Lie group. A ***Lie subgroup*** of G is a subgroup $H \subseteq G$ endowed with a topology and a smooth structure making it into both a Lie group and an immersed submanifold of G. In this section we will describe some basic examples of Lie subgroups.

Fact 5 (Embedded subgroups are Lie subgroups)**.** *Let G be a Lie group and $H \subseteq G$ a subgroup that is also an embedded submanifold. Then H is a Lie subgroup of G.*

Proof. Since an embedded submanifold is trivially an immersed submanifold, we already know that H is a subgroup of G and an immersed submanifold of G. We just need to check that H is a Lie group in its own right, i.e. that the multiplication and inversion operations in H are smooth. The inclusion $i : H \times H \hookrightarrow G \times G$ is a smooth embedding because H is an embedded submanifold of G. Thus the multiplication in H is obtained by restricting the multiplication $m : G \times G \to G$ in G to H,

$$m \circ i : H \times H \overset{i}{\hookrightarrow} G \times G \xrightarrow{m} H$$

where we have landed in H because H is a subgroup. Thus the multiplication in H is smooth as a composition of smooth maps. Similarly, the inversion map in H is smooth as a composition of the (smooth) inversion in G with the smooth embedding $H \hookrightarrow G$. ∎

The simplest example of embedded Lie subgroups are the open subgroups. However, as the following fact demonstrates, the collection of open subgroups of a Lie group is very limited.

Fact 6. *Let G be a Lie group. Any open subgroup $H \subseteq G$ is also closed, hence a union of connected components of G.*

Proof. Every left coset gH is an open subset of G as the image of the open set H under the diffeomorphism L_g. Hence the complement is open as the union of open cosets, $G \setminus H = \bigcup_{g \notin H} gH$, which means that H is closed. Since H is both open and closed it must be a union of connected components of G. ∎

For any subset $W \subseteq G$, let $\langle W \rangle$ denote the subgroup generated by W. Recall that this subgroup can be understood as the intersection of all subgroups of G containing W, or equivalently as the set

$$\langle W \rangle = \{w_1 w_2 \cdots w_m : m \geq 0 \text{ and } w_i \in W \cup W^{-1}\}$$

consisting of all words in G obtained by concatenating elements of W and their inverses.

Fact 7 (Neighborhoods of the identity in a Lie group)**.** *Let G be a Lie group and let $W \subseteq G$ be any open neighborhood of $e \in G$. Then*

(i) W generates an open subgroup $\langle W \rangle$ of G.

(ii) If W is connected then it generates a connected open subgroup of G.

(iii) If G is connected then W generates all of G.

Proof. For any subsets $A, B \subseteq G$ we adopt the notation

$$AB = \{ab : a \in A, b \in B\}$$
$$A^{-1} = \{a^{-1} : a \in A\}$$

(i) For any $k \geq 1$, let W_k denote the set of all elements in G that can be expressed as a product of k or fewer elements of $W \cup W^{-1}$. Note that $W^{-1} = j(W)$ is open as the image of an open set under the diffeomorphism j, hence $W_1 = W \cup W^{-1}$ is open. For each $k > 1$ we then have

$$W_k = W_1 W_{k-1} = \bigcup_{g \in W_1} L_g(W_{k-1})$$

and since L_g is a diffeomorphism it follows by induction that each W_k is open as a union of open sets. Therefore $\langle W \rangle = \bigcup_{k \geq 1} W_k$ is open.

(ii) Suppose $W \subseteq G$ is connected. Then $W^{-1} = j(W)$ is connected as the image of a connected set under the diffeomorphism j, and so $W_1 = W \cup W^{-1}$ is connected as a union of connected sets having nonempty intersection (they the identity in common). Then $W_2 = m(W_1 \times W_1)$ is connected as the image of a connected set under the continuous map m, and it follows by induction that $W_k = m(W_1 \times W_{k-1})$ is connected for each $k \geq 1$. Therefore $\langle W \rangle = \bigcup_{k \geq 1} W_k$ is connected as a union of connected sets having the identity in common.

(iii) Suppose G is connected. By (ii), we know that $\langle W \rangle$ is an open subgroup of G, and by Fact 6 it's also a closed subgroup. But G is connected so we must have $\langle W \rangle = G$.

■

Let G be a Lie group. The connected component of G containing the identity $e \in G$ is an open neighborhood of the identity called the ***identity component*** of G, and we denote it by $G_0 \subseteq G$. The following fact shows that, in some sense, this is the only connected component of G that we need to study.

Fact 8 (Identity component of a Lie group)**.** *Let G be a Lie group with identity component $G_0 \subseteq G$. Then*

(i) G_0 is a normal subgroup of G.

(ii) G_0 is the only connected open subgroup of G.

(iii) Every connected component of G is diffeomorphic to G.

Proof. (i) By Fact 7 the identity component G_0 generates a connected open subgroup of G, so $\langle G_0 \rangle$ is connected and is also a union of connected components of G. Hence $\langle G_0 \rangle$ must itself be a connected component of G, and the only possibility is that $\langle G_0 \rangle = G_0$ since it contains the identity. This shows that G_0 is a connected open subgroup of G. Let $g \in G$. Then the conjugation map $C_g : G \to G$ given by $x \mapsto gxg^{-1}$ is a diffeomorphism mapping G_0 into a connected subset of G containing the identity (since $C_g(e) = e$). Thus $C_g(G_0) \subseteq G_0$ which means that G_0 is normal.

(ii) Any connected open subgroup of G is a connected open neighborhood of $e \in G$, and by Fact 7 it generates a connected open subgroup of G. The same argument as before shows that it must be G_0.

(iii) Let $H \subseteq G$ be any connected component of G. For any $h \in H$, $L_h(G_0) = hG_0$ is a connected subset of H which is diffeomorphic to G_0. In fact since G_0 is both open and closed, hG_0 must also be both open and closed, which implies that $hG_0 = H$ and so H is diffeomorphic to G_0.

■

Fact 6 shows that open subgroups are all unions of connected components, and by Fact 8 the identity component tells us everything about the other components. So we essentially know what all of the open subgroups look like. Next we will discuss some useful ways of producing many more embedded Lie subgroups (not just open ones).

Fact 9. *Let $F : G \to H$ be a Lie group homomorphism. Then* $\ker F$ *is an embedded Lie subgroup of G with* $\dim \ker F = \dim G - \operatorname{rank} F$*; i.e.* $\operatorname{codim} \ker F = \operatorname{rank} F$.

Proof. Since F has constant rank by Fact 4 it follows from the constant-rank level set theorem that $\ker F = F^{-1}(e)$ is an embedded submanifold with $\operatorname{codim} \ker F = \operatorname{rank} F$. The kernel of F is a subgroup of G for algebraic reasons, hence by Fact 5 the kernel is also a Lie subgroup. ■

Fact 10. *Let $F : G \to H$ be an injective Lie group homomorphism. Then the image $F(G)$ has a unique smooth manifold structure such that $F(G)$ is a Lie subgroup of H and $F : G \to F(G)$ is a Lie group isomorphism.*

Proof. Since F is injective and constant rank, it must be a smooth immersion by the global rank theorem. Thus $F(G)$ has a unique smooth structure making it into an immersed submanifold of H such that $F : G \to F(G)$ is a diffeomorphism (and also an isomorphism of groups). Moreover, $F(G)$ is a subgroup of H for algebraic reasons, so all that remains is to show that $F(G)$ is a Lie group in its own right; i.e. that it has smooth multiplication and inversion operations. Let $\widehat{m} : F(G) \times F(G) \to F(G)$ denote multiplication in $F(G)$. Then for any $g, h \in G$ we have

$$\widehat{m}(F(g), F(h)) = F(g)F(h) = F(gh) = F(m(g, h))$$

so $\widehat{m} \circ (F \times F) = F \circ m$. But F is a diffeomorphism onto its image so

$$\widehat{m} = (F \circ m) \circ (F \times F)^{-1}$$

hence $\widehat{m}$ is smooth as a composition of smooth maps. A similar argument shows that the inversion map $\widehat{i} : F(G) \to F(G)$ is smooth. ■

We pause here to observe an interesting aspect of the preceding fact. Even though a subset of a Lie group which is both a subgroup and immersed submanifold need not be a Lie group itself (i.e. the restricted operations need not be smooth), we were able to use the group homomorphism F to express the multiplication $\widehat{m}$ in $F(G)$ as a composition of smooth maps without needing the inclusion $F(G) \hookrightarrow H$ to be smooth embedding.

Example 3 (Embedded Lie subgroups)**.**

(a) The subgroup $\mathrm{GL}^+(n, \mathbb{R}) \subseteq \mathrm{GL}(n, \mathbb{R})$ is an open subgroup of $\mathrm{GL}(n, \mathbb{R})$, hence an embedded Lie subgroup.

(b) S^1 is a subgroup of $\mathbb{C}^\times$, and also an embedded submanifold, so it's an embedded Lie subgroup by Fact 5.

(c) The real special linear group $\mathrm{SL}(n,\mathbb{R})$ is an embedded Lie subgroup of $\mathrm{GL}(n,\mathbb{R})$ by Fact 9 because it's the kernel of the Lie group homomorphism $\det : \mathrm{GL}(n,\mathbb{R}) \to \mathbb{R}^\times$. Moreover, the determinant function is surjective, so it's a smooth submersion by the global rank theorem, and therefore $\dim \mathrm{SL}(n,\mathbb{R}) = \dim \mathrm{GL}(n,\mathbb{R}) - \dim \mathbb{R}^\times = n^2 - 1$.

(d) Define a map $\beta : \mathrm{GL}(n,\mathbb{C}) \to \mathrm{GL}(2n,\mathbb{R})$ by replacing each complex entry $a + ib$ with the 2×2 block $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$. This is an injective Lie group homomorphism, whose image is an embedded Lie subgroup of $\mathrm{GL}(2n,\mathbb{R})$. Thus, $\mathrm{GL}(n,\mathbb{C})$ is isomorphic to this Lie subgroup of $\mathrm{GL}(2n,\mathbb{R})$.

(e) The complex special linear group $\mathrm{SL}(n,\mathbb{C}) \subseteq \mathrm{GL}(n,\mathbb{C})$ is the kernel of the Lie group homomorphism $\det : \mathrm{GL}(n,\mathbb{C}) \to \mathbb{C}^\times$. Since the determinant homomorphism is surjective, it's a smooth submersion by the global rank theorem, and therefore $\dim \mathrm{SL}(n,\mathbb{C}) = \dim \mathrm{GL}(n,\mathbb{C}) - \dim \mathbb{C}^\times = 2n^2 - 2$.

Example 4 (Lie subgroup that is not embedded)**.** Let α be an irrational number and let $\gamma : \mathbb{R} \to \mathbb{T}^2 \subseteq \mathbb{C}^2$ be the curve on the 2-torus defined by $\gamma(t) = (e^{2\pi i}t, e^{2\pi i\alpha t})$. Then γ is an injective Lie group homomorphism (hence an injective smooth immersion), so its image $\gamma(\mathbb{R}) \subseteq \mathbb{T}^2$ is a 1-dimensional immersed submanifold of $\mathbb{T}^2$; however, $\gamma(\mathbb{R})$ is not an embedded submanifold of $\mathbb{T}^2$. The image $\gamma(\mathbb{R})$ is dense in $\mathbb{T}^2$, so if we try to use the subspace topology naively to select an open disk (in $\mathbb{T}^2$) around some point in $\gamma(\mathbb{R})$, then it will contain infinitely many path components: certainly not homeomorphic to $\mathbb{R}^4$!

Of course, $\gamma(\mathbb{R})$ is equipped with its own topology (not the subspace topology) making it into a 1-dimensional topological manifold (by taking open intervals along the curve, for example). In summary, $\gamma(\mathbb{R})$ is a Lie subgroup of $\mathbb{T}^2$ but not an embedded Lie subgroup.

In general, a smooth submanifold can be closed but not embedded, or embedded but not closed; for example, the figure eight curve in $\mathbb{R}^2$ is closed but not embedded, and the open unit ball in $\mathbb{R}^n$ is embedded but not closed. For Lie groups, on the other hand, there is a close relationship between being closed and being embedded – in fact, a subgroup $H \subseteq G$ is an embedded Lie subgroup if and only if it's topologically closed! We will forego the proof (because a proper proof involves the machinery of Lie algebras and exponential maps) but it's such a crucial theorem that we will state it here:

Theorem 5 (Closed subgroup theorem)**.** *Let G be a Lie group and let $H \subseteq G$ be a subgroup. The following are equivalent:*

(i) H is an embedded Lie subgroup of G.

(ii) H is an embedded submanifold of G.

(iii) H is closed in G.

Corollary 2. *Let G be a Lie group. For any subgroup $H \subseteq G$, the closure $\overline{H}$ is an embedded Lie subgroup of G.*

As an immediate consequence, we deduce that all Lie subgroups are either embedded, or they are dense subgroups of embedded Lie subgroups. The prototypical example: the image of the irrational line on the torus under the embedding

$\mathbb{T}^2 \hookrightarrow \mathbb{T}^3$. It's a Lie subgroup of $\mathbb{T}^3$ which is not closed, embedded, or dense, but it's obviously dense in its closure, which is an embedded Lie subgroup of $\mathbb{T}^3$.

4 Lie group actions

Let G be a group and M any set. A ***left action*** of G on M, or a ***left G-action*** on M, is a map $\theta : G \times M \to M$, written as $\theta(g, x) = g \cdot x$, satisfying

$$\begin{aligned}
&g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x, \\
&e \cdot x = x,
\end{aligned}$$

for every $g_1, g_2 \in G$ and $x \in M$. A right action of G on M is defined analogously.

If G is a topological group and M is a topological space, a (left or right) action of G on M is called a ***continuous action*** if the defining map $G \times M \to M$ or $M \times G \to G$ is continuous; similarly, if G is a Lie group acting on a smooth manifold M then the action of G on M is called a ***smooth action*** if the defining map is smooth. In this note we will be primarily concerned with Lie groups acting smoothly on smooth manifolds, and unless otherwise stated we will assume that the group acts on the left.

Here's another useful way of thinking about group actions. To say that a topological group G acts continuously on a topological manifold M is equivalent to saying that for any fixed $g \in G$ we have a homeomorphism $\theta_g : M \to M$ defined by $\theta_g(x) = g \cdot x$; indeed, just observe directly from the definition that $\theta_{g^{-1}}$ is the inverse of θ_g, and this inverse is also continuous. Therefore, a continuous group action $\theta : G \times M \to M$ is the same as a family of homeomorphisms of M $\{\theta_g\}_{g \in G}$ indexed by G. In the same way, if G is a Lie group and M is a smooth manifold then a smooth group action $\theta : G \times M \to M$ is the same as a family of diffeomorphisms of M indexed by G.

Let $\theta : G \times M \to M$ be a G-action on M. We adopt the following standard terminology:

- For any $p \in M$, the ***orbit*** of p is the set

$$G \cdot p = \{g \cdot p : g \in G\} \subseteq M$$

  consisting of the points in M that can be reached by permuting p.

- For any $p \in M$, the ***stabilizer*** of p is the set

$$G_p - \{g \in G : g \cdot p = p\} \subseteq G$$

  consisting of the group elements which fix p. It follows from the definition of a group action that the stabilizer G_p is always a subgroup of G.

- The G-action is ***transitive*** if, for every $p, q \in M$, there exists a $g \in G$ such that $g \cdot p = q$. In other words, $G \cdot p = M$ for every $p \in M$, so M is the only orbit.

- The G-action is ***free*** if $G_p = \{e\}$ for every $p \in M$; i.e. the only element of G that fixes anything in M is the identity.

Example 5 (Lie group actions on smooth manifolds)**.**

(a) Let G be any Lie group and M any smooth manifold. The trivial action of G on M given by $g \cdot p = p$ for every $g \in G$ and $p \in M$ is a smooth action for which every orbit is a single point – i.e. $G \cdot p = p$ – and each stabilizer is all of G – i.e. $G_p = G$.

(b) The general linear group $\mathrm{GL}(n, \mathbb{R})$ acts naturally on $\mathbb{R}^n$ via matrix multiplication $(A, x) \mapsto Ax$. This is a smooth action because the components of Ax are polynomials in the entries of A and x. There are exactly two orbits, $\{0\}$ and $\mathbb{R}^n \setminus \{0\}$, because any nonzero vector can be taken to any other by some invertible linear transformation (just perform some combination of rotations and scaling).

(c) Every Lie group G acts smoothly on itself by left translation, $(g, h) \mapsto gh$. Evidently this action is both free and transitive.

(d) Every Lie group G acts smoothly on itself by conjugation, $(g, h) \mapsto ghg^{-1}$. The stabilizer of $h \in G$ is exactly the centralizer of h, i.e. the set of elements which commute with h. The orbits of this action are precisely the conjugacy classes of G.

(e) $\mathbb{Z}^n$ acts on $\mathbb{R}^n$ smoothly and freely by translation.

The machinery of group actions establishes a nice condition that in many cases allows us to easily prove that a smooth map has constant rank. Suppose we have a group G acting on two sets M and N. We say that $F : M \to N$ is ***equivariant*** with respect to the G-actions if

$$\begin{aligned} F(g \cdot p) &= g \cdot F(p) \qquad \text{(for } G \text{ acting on the left)} \\ F(p \cdot g) &= F(p) \cdot g \qquad \text{(for } G \text{ acting on the right)} \end{aligned}$$

holds for every $g \in G$ and $p \in M$. Equivalently, in terms of the families of bijections $\theta_g : M \to M$ and $\varphi_g : N \to N$, this condition means that

$$\varphi_g \circ F = F \circ \theta_g$$

for every $g \in G$.

$$\begin{tikzcd} M \arrow[r, "F"] \arrow[d, "\theta"'] & N \arrow[d, "\varphi"] \\ M \arrow[r, "F"'] & N \end{tikzcd}$$

Theorem 6 (Equivariant rank theorem)**.** *Let G be a Lie group acting smoothly on two smooth manifolds M and N, and let $F : M \to N$ be a smooth map. Suppose that*

- *G acts transitively on M.*
- *F is equivariant with respect to the G-actions.*

Then F has constant rank.

Proof. This proof follows the same pattern as Fact 4. Let $\theta : G \times M \to M$ denote the transitive G-action on M and let $\varphi : G \times N \to N$ denote the G-action on N.

Fix two points $p, q \in M$. Since G acts transitively on M we can find some $g \in G$ such that $\theta_g(p) = q$, and since F is equivariant we have

$$\varphi_g \circ F = F \circ \theta_g$$

so taking the differential at p on both sides yields

$$d(\varphi_g)_{F(p)} \circ dF_p = dF_q \circ d(\theta_g)_p.$$

But θ_g and φ_g are diffeomorphisms, so their differentials are isomorphisms, and therefore

$$dF_q = d(\varphi_g)_{F(p)} \circ dF_p \circ d(\theta_g)_p^{-1}$$

which means that F has the same rank at p and q, and F has constant rank. ∎

We motivate the first application of the equivariant rank theorem by appealing to some basic results from combinatorial group theory. Suppose we have a group G acting on a set M. For each $p \in M$, define the ***orbit map*** $\theta^{(p)} : G \to M$ by $\theta^{(p)}(g) = g \cdot p$ (kind of like "evaluation at p"). Notice that the image of $\theta^{(p)}$ is

$$\theta^{(p)}(G) = G \cdot p = \text{ orbit of } p \text{ under } G,$$

and the fiber of p under $\theta^{(p)}$ is

$$\left(\theta^{(p)}\right)^{-1}(p) = G_p = \text{ stabilizer of } p \text{ in } G.$$

Observe that, for any $p \in M$, we can partition G into cosets of G_p (each of size $|G_p|$) to get a bijection $G/G_p \longleftrightarrow G \cdot p$, and this yields a formula

$$|G| = |G \cdot p||G_p|$$

which is the so-called orbit-stabilizer theorem. Thus, even in the context of sets, the orbit map already provides useful combinatorial information about the group G and the set M. It should come as no surprise, then, that the orbit map provides even more useful information in the context of smooth manifolds.

Fact 11 (Properties of the orbit map)**.** *Let M be a smooth manifold and let θ be a smooth G-action on M. Then for each $p \in M$:*

(i) The orbit map $\theta^{(p)}$ is smooth and constant rank, so the stabilizer G_p is an embedded Lie subgroup of G.

(ii) If $G_p = \{e\}$, then $\theta^{(p)}$ is an injective smooth immersion, and so the orbit $G \cdot p$ is an immersed submanifold of M.

Proof. (i) First note that $\theta^{(p)}$ is smooth as a composition of smooth maps:

$$\theta^{(p)} = \theta \circ i : G \simeq G \times \{p\} \xhookrightarrow{i} G \times M \xrightarrow{\theta} M.$$

In order to show $\theta^{(p)}$ has constant rank it suffices to show that it's equivariant with respect to some suitable G-actions. Indeed, G acts transitively on itself by left multiplication, and for any $g, g' \in G$ we have

$$\begin{aligned} \theta^{(p)}(g' \cdot g) &= \theta^{(p)}(g'g) \\ &= (g'g) \cdot p \\ &= g' \cdot (g \cdot p) \\ &= g' \cdot \left(\theta^{(p)}(g)\right) \end{aligned}$$

and so $\theta^{(p)}$ has constant rank by the equivariant rank theorem. Finally, we know that fibers of constant rank smooth maps are embedded submanifolds, so the stabilizer $G_p = \left(\theta^{(p)}\right)^{-1}(p) \subseteq G$ is an embedded submanifold of G, and it's a subgroup for algebraic reasons. Hence G_p is a Lie subgroup by Fact 5.

(ii) Suppose $G_p = \{e\}$. If $\theta^{(p)}(g) = \theta^{(p)}(g')$ then $g' \cdot p = g \cdot p$ implies that $(g^{-1}g') \cdot p = e$ so $g^{-1}g' \in G_p$ and therefore $g^{-1}g' = e$. Hence $g' = g$ and $\theta^{(p)}$ is injective. Since $\theta^{(p)}$ also has constant rank, it follows from the global rank theorem that $\theta^{(p)}$ is a smooth immersion, so the orbit $G \cdot p$ is an immersed submanifold of M as the image of a smooth immersion.

■

The matrix groups provide a good demonstration of how the equivariant rank theorem can be used in practice.

Example 6 (The orthogonal group)**.** Recall that a matrix $A \in M(n, \mathbb{R})$ is orthogonal if its columns constitute an orthonormal basis for $\mathbb{R}^n$, or equivalently if $A^T A = I_n$. Let $O(n)$ denote the group of $n \times n$ orthogonal matrices. In this example we will establish the following important basic facts about $O(n)$:

- $O(n)$ is an embedded Lie subgroup of $\mathrm{GL}(n, \mathbb{R})$.
- The dimension of $O(n)$ as a submanifold of $\mathrm{GL}(n, \mathbb{R})$ is $n(n-1)/2$.
- $O(n)$ is compact.

Define $\phi : \mathrm{GL}(n, \mathbb{R}) \to M(n, \mathbb{R})$ by $\phi(A) = A^T A$. Then $O(n) = \phi^{-1}(I_n)$, so in order to show that $O(n)$ is an embedded Lie subgroup it will suffice to show that ϕ has constant rank. Define a right action of $\mathrm{GL}(n, \mathbb{R})$ on itself by right multiplication:

$$B \cdot A = BA \qquad \text{for every } A, B \in \mathrm{GL}(n, \mathbb{R}).$$

Moreover, $\mathrm{GL}(n, \mathbb{R})$ acts on $M(n, \mathbb{R})$ on the right by:

$$X \cdot A = A^T X A \qquad \text{for every } A \in \mathrm{GL}(n, \mathbb{R}) \text{ and } X \in M(n, \mathbb{R}).$$

These are smooth actions and ϕ is equivariant with respect to them because

$$\begin{aligned} \phi(B \cdot A) &= \phi(BA) \\ &= (BA)^T BA \\ &= A^T B^T BA \\ &= (B^T B) \cdot A \\ &= \phi(B) \cdot A \end{aligned}$$

so ϕ has constant rank by the equivariant rank theorem, and therefore $O(n)$ is an embedded Lie subgroup of $\mathrm{GL}(n, \mathbb{R})$ (by combining the constant-rank level set theorem with Fact 5).

Note that $O(n)$ is a closed subset of $M(n, \mathbb{R})$ as the pre-image of the closed set $\{I_n\}$ under the continuous map ϕ. Furthermore, every orthogonal matrix has norm equal to $\sqrt{n}$ because the columns of an orthogonal matrix are unit vectors. Hence $O(n)$ is a closed and bounded subset of $M(n, \mathbb{R}) \simeq \mathbb{R}^{n^2}$, which means it's compact.

Now what is the dimension of $O(n)$ as a submanifold of $\mathrm{GL}(n,\mathbb{R})$? Note that

$$\dim O(n) = \dim \phi^{-1}(I_n) = n^2 - \operatorname{rank}\phi = n^2 - \dim \operatorname{im} d\phi_{I_n}$$

by the constant-rank level set theorem, so we just need to analyze the differential $d\phi_{I_n}$. For any $B \in T_{I_n}GL(n,\mathbb{R}) \simeq M(n,\mathbb{R})$ we can calculate $d\phi_{I_n}(B)$ using Fact 1 by choosing any smooth curve γ in $M(n,\mathbb{R})$ with $\gamma(0) = I_n$ and $\gamma'(0) = B$. Evidently $\gamma(t) = I_n + tB$ works. Thus we calculate

$$(\phi \circ \gamma)(t) = (I_n + tB)^T(I_n + tB) = I_n + t(B + B^T) + t^2 B^T B,$$

from which we can immediately identify the derivative as the linear term $B + B^T$. Therefore

$$d\phi_{I_n}(B) = (\phi \circ \gamma)'(0) = B + B^T$$

for any $B \in M(n,\mathbb{R})$. This expression makes it clear that $\operatorname{im} d\phi_{I_n}$ is contained in the group of symmetric $n \times n$ matrices, but in fact for any symmetric matrix $B \in M(n,\mathbb{R})$ we have

$$d\phi_{I_n}\left(\frac{1}{2}B\right) = \frac{1}{2}(B + B^T) = B$$

so $\operatorname{im} d\phi_{I_n}$ is exactly the group of symmetric $n \times n$ matrices. As a result, the dimension of the image of the differential is

$$\dim \operatorname{im} d\phi_{I_n} = n(n+1)/2$$

since any symmetric matrix is completely determined by the $n(n+1)/2$ entries on and above the diagonal. Therefore

$$\dim O(n) = n^2 - \dim \operatorname{im} d\phi_{I_n} = n^2 - n(n+1)/2 = n(n-1)/2.$$

Summary (Constructing embedded Lie subgroups)**.** With the orthogonal group as our paradigmatic example, let's summarize the process for constructing embedded Lie subgroups. Suppose we have a smooth manifold M and we want to construct an embedded submanifold $S \subseteq M$, or prove that a given subset is an embedded submanifold. One method is as follows:

1. Find a Lie group G acting smoothly on M and N, such that the G-action on M is transitive.
2. Construct a G-equivariant smooth map $\phi : M \to N$ with $S = \phi^{-1}(p)$ for some $p \in N$.
3. Then the equivariant rank theorem implies that ϕ has constant rank, so by the constant rank level set theorem it follows that $S = \phi^{-1}(p)$ is an embedded submanifold of M. Moreover, we have $\dim S = \dim M - \operatorname{rank}\phi$.
4. Additionally, if M is a Lie group and S is a subgroup, then S is an embedded Lie subgroup by Fact 5.

5 Quotient topology

The goal for the next part of this note is to investigate the following important question: given a Lie group G acting smoothly on a smooth manifold M, when is the orbit space M/G also a smooth manifold? In order to state and prove some theorems regarding this question, we will need to know some basic facts about the topology on the set of orbits M/G (i.e. the quotient topology). The purpose of this section is to collect these preliminary facts before moving on to the smooth structure of this set, i.e. the subject of quotient manifolds.

Let X be a topological space and $\sim$ an equivalence relation on X. Denote the resulting set of equivalence classes by $X/\!\sim$, and let $q : X \to X/\!\sim$ denote the canonical projection. The definition of the quotient topology is motivated by our desire to put a suitable topology on the set $X/\!\sim$, turning it into a *quotient space*. Inspired by the characteristic property which defines such topologies as the subspace topology, product topology, etc., we might demand that any $f : X/\!\sim \to Y$ be continuous if and only if $f \circ q : X \to Y$ is continuous. This turns out to be equivalent to the demand that $U \subseteq X/\!\sim$ be open if and only if $q^{-1}(U) \subseteq X$ is open. This is precisely the quotient topology on $X/\!\sim$.

$$\begin{tikzcd} X \arrow[d, "q"'] \arrow[dr, "f\circ q"] & \\ X/\!\sim \arrow[r, "f"'] & Y \end{tikzcd}$$

More generally: suppose we have two topological spaces X and Y and a surjection $q : X \to Y$ (thinking of the previous situation where q is the canonical projection onto $X/\!\sim$). We say that q is a ***quotient map*** if it has the property that: $U \subseteq Y$ is open in Y if and only if $q^{-1}(U)$ is open in X. Notice that one direction of this property is exactly continuity, so in particular quotient maps are continuous.

Fact 12. *Let X be a topological space and Y a set. If $q : X \to Y$ is a surjective function, then there exists exactly one topology on Y for which q is a quotient map. This topology is called the* ***quotient topology*** *on Y.*

Proof. By analogy with our description of the topology on $X/\!\sim$, we define a topology on Y by saying that $U \subseteq Y$ is open if and only if $q^{-1}(U) \subseteq X$ is open. It's straightforward to check that this indeed defines a topology on Y. Obviously q is a quotient map with respect to this topology. We just need to show that this is the unique topology for which q is a quotient map. Call the aforementioned topology $\mathcal{T}$ and suppose that $\mathcal{T}'$ is another topology on Y for which q is a quotient map. Then we have $U \in \mathcal{T}'$ if and only if $q^{-1}(U)$ is open in X if and only if $U \in \mathcal{T}$, so $\mathcal{T} = \mathcal{T}'$. ∎

Thus, given a surjective map $q : X \to Y$, the quotient topology on Y is the unique topology which turns q into a quotient map. In this case, equipping Y with the quotient topology, we say that Y is a ***quotient of* X** or simply call Y a ***quotient space***. In light of the previous fact, saying that $q : X \to Y$ is a quotient map automatically implies that Y is equipped with the quotient topology.

Fact 13 (Characteristic property of the quotient topology)**.** *Let $q : X \to Y$ be a quotient map, Z any topological space and $f : Y \to Z$ any function. Then f is continuous if and only if $f \circ q : X \to Z$ is continuous.*

Suppose we have a quotient map $q : X \to Y$ and a continuous map $g : X \to Z$. Since Y is a quotient of X, it's natural to wonder when it's possible to complete the diagram by passing from g to a continuous map $\widetilde{g} : Y \to Z$. In the event that such a completion is possible, we say that "g descends to the quotient" via $\widetilde{g}$. We have the following useful characterization:

Fact 14 (Descending to the quotient)**.** *Let $q : X \to Y$ be a quotient map and $g : X \to Z$ a continuous map. Then the following are equivalent:*

(i) There exists a unique continuous map $\widetilde{g} : Y \to Z$ such that $g = \widetilde{g} \circ q$.

(ii) $q(x_1) = q(x_2) \implies g(x_1) = g(x_2)$ for every $x_1, x_2 \in X$.

Moreover, the map $\widetilde{g}$ is injective if and only if $q(x_1) = q(x_2) \iff g(x_1) = g(x_2)$ holds.

Proof. First suppose that there exists a map $\widetilde{g}$ such that $g = \widetilde{g} \circ q$. Note that, by the characteristic property in Fact 13, $\widetilde{g}$ is continuous because g is continuous. If $q(x_1) = q(x_2)$, then $\widetilde{g}(q(x_1)) = \widetilde{g}(q(x_2))$, hence $g(x_1) = g(x_2)$.

Conversely, suppose that (b) holds. Let $y \in Y$. Since q is surjective, there exists an $x \in X$ such that $y = q(x)$. Define $\widetilde{g} : Y \to Z$ by $\widetilde{g}(y) = g(x)$. Then $\widetilde{g}$ is well-defined because if $y_1 = q(x_1)$ and $y_2 = q(x_2)$ are such that $y_1 = y_2$, then $q(x_1) = q(x_2)$ and this implies by assumption (b) that $g(x_1) = g(x_2)$. Hence by our definition of $\widetilde{g}$ we have $\widetilde{g}(y_1) = g(x_1) = g(x_2) = \widetilde{g}(y_2)$. This shows that $\widetilde{g}$ is well-defined. Now for every $y \in Y$ if $y = q(x)$ then $\widetilde{g}(q(x)) = \widetilde{g}(y) = g(x)$ so $\widetilde{g} \circ q = g$.

For the last statement, suppose that the map $\widetilde{g}$ exists. If $\widetilde{g}$ is injective and $g(x_1) = g(x_2)$ then $\widetilde{g}(q(x_1)) = \widetilde{g}(q(x_2))$, but since $\widetilde{g}$ is injective we have $q(x_1) = q(x_2)$, hence we have shown the converse of (b) as desired. On the other hand, suppose that $q(x_1) = q(x_2) \iff g(x_1) = g(x_2)$. Let $\widetilde{g}(y_1) = \widetilde{g}(y_2)$ where $y_1 = q(x_1)$ and $y_2 = q(x_2)$. Then $\widetilde{g}(q(x_1)) = \widetilde{g}(q(x_2))$. We want to show that $\widetilde{g}$ is injective. Since $\widetilde{g} \circ q = g$ we have $g(x_1) = g(x_2)$ but then it follows (from our assumption) that $q(x_1) = q(x_2)$ and hence $y_1 = y_2$. Thus $\widetilde{g}$ is injective. ■

$$
\begin{tikzcd}
X \arrow[r, "g"] \arrow[d, "q"'] & Z \\
Y \arrow[ur, "\widetilde{g}"'] &
\end{tikzcd}
$$

Condition (ii) in Fact 14 can be summarized by saying that "g is constant on the fibers of q", and the relation in the last line of Fact 14, "$q(x_1) = q(x_2) \iff g(x_1) = g(x_2)$ for every $x_1, x_2 \in X$," is summarized by the statement that "q and g make the same identifications". Thus with this terminology, Fact 14 says that:

- g descends to the quotient via $\widetilde{g}$ if and only if g is constant on the fibers of q.
- The induced map $\widetilde{g}$ on the quotient is injective if and only if g and q make the same identifications.

We can use Fact 14 to prove the following theorem, giving us a nice condition for determining that two quotients of the same space X are homeomorphic.

Theorem 7. *Let $q : X \to Y$ and $p : X \to Z$ be quotient maps. Suppose that p and q make the same identifications; that is,*

$$p(x_1) = p(x_2) \iff q(x_1) = q(x_2)$$

for every $x_1, x_2 \in X$. Then Y and Z are homeomorphic.

Proof. By Fact 14 there exists a unique continuous map $\widetilde{p} : Y \to Z$ such that $\widetilde{p} \circ q = p$. We will show that $\widetilde{p}$ is a homeomorphism. Also by Fact 14, $\widetilde{p}$ is injective. Since p is surjective, $\widetilde{p}$ must also be surjective. Thus $\widetilde{p}$ is a continuous bijection.

$$\begin{tikzcd} X \arrow[r, "p"] \arrow[d, "q"'] & Y \\ Z \arrow[ur, "\tilde{p}"'] & \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "q"] \arrow[d, "p"'] & Z \\ Y \arrow[ur, "\tilde{q}"'] & \end{tikzcd}$$

Since p and q are *both* quotient maps, we can apply Fact 14 again with the roles of p and q reversed to get the diagram on the right and the unique continuous bijection $\widetilde{q} : Z \to Y$ such that $\widetilde{q} \circ p = q$. Composing both sides of this equation with $\widetilde{p}$ on the left, we get $\widetilde{p} \circ \widetilde{q} \circ p = \widetilde{p} \circ q = p$. Hence $\widetilde{p} \circ \widetilde{q} = \mathrm{Id}_Z$ is the identity map on Z. Therefore $\widetilde{q}$ is the unique right inverse of $\widetilde{p}$. Hence $(\widetilde{p})^{-1} = \widetilde{q}$, so the inverse of $\widetilde{p}$ is also continuous. Thus $\widetilde{p}$ is a homeomorphism. ■

The following corollary tells us that every quotient Y of a topological space X really is a space of equivalence classes, in the sense that it is homeomorphic to $X/{\sim}$ with the relation $\sim$ which identifies two points in X if they lie in the same fiber of the quotient map q.

Corollary 3. *Let $q : X \to Y$ be a quotient map. Letting $\sim$ denote the equivalence relation on X defined by $x_1 \sim x_2$ if and only if $q(x_1) = q(x_2)$, we have $Y \simeq X/{\sim}$.*

Proof. Let $p : X \to X/{\sim}$ be the canonical projection $p(x) = [x]$. It's straightforward to verify that p and q then satisfy

$$p(x_1) = p(x_2) \iff q(x_1) = q(x_2)$$

for every $x_1, x_2 \in X$, so applying Theorem 7 with $Z = X/{\sim}$ gives the desired homeomorphism. ■

When is a quotient space Hausdorff? Recall the following basic fact from point-set topology: a topological space X is Hausdorff if and only if the diagonal $\Delta = \{(x, x) : x \in X\} \subseteq X \times X$ is closed with respect to the product topology. There is a similar result for quotient spaces which we will now discuss. Let $q : X \to Y$ be a quotient map and consider the set

$$R = \{(x_1, x_2) : q(x_1) = q(x_2)\} \subseteq X \times X.$$

Observe that $R = (q \times q)^{-1}(\Delta)$, so if Y is Hausdorff then Δ is closed in $Y \times Y$ and therefore R is closed in $X \times X$; hence, a necessary condition for a quotient of X to be Hausdorff is that the set R be closed in $X \times X$. In fact, the converse is also true provided that q is an open map or alternatively that X is compact and Hausdorff.

Fact 15 (Hausdorff quotient space)**.** *Let $q : X \to Y$ be a quotient map, and suppose that one of the following holds:*

(i) q is an open map.

(ii) X is a compact Hausdorff space.

Then Y is Hausdorff if and only if $R \subseteq X \times X$ is closed.

Proof. In each case it will suffice to prove the "if" direction because the other direction has already been proved. Thus, we assume that R is closed and we aim to prove that Y is Hausdorff.

First suppose that q is an open map. Then $q \times q$ is an open map, and we need only observe that the diagonal $\Delta \subseteq Y \times Y$ is closed, because its complement is (using the surjectivity of q)

$$\Delta^c = \{(q(x_1), q(x_2)) : q(x_1) \neq q(x_2)\} = (q \times q)(R^c)$$

which is open as the image of the open set R^c under $q \times q$.

Now suppose that X is a compact Hausdorff space. We will show that q is a closed map. Let $p_1, p_2 : X \times X \to X$ denote the projections onto the first and second factors, respectively. Note that the projections are closed maps because X is compact Hausdorff. Taking any closed set $C \subseteq X$, we have

$$\begin{aligned} q^{-1}(q(C)) &= \{x \in X : q(x) \in q(C)\} \\ &= \{x \in X : q(x) = q(c) \text{ for some } c \in C\} \\ &= \{x \in X : (x, c) \in R \text{ for some } c \in C\} \\ &= p_1(p_2^{-1}(C) \cap R) \end{aligned}$$

which is closed because R is closed, p_2 is continuous, and p_1 is closed. Thus $q(C)$ is closed by definition of the quotient topology on Y, and q is a closed map. As a result, every point $\{q(x)\}$ is closed in Y. Take any two distinct points $q(a)$ and $q(b)$ in Y. Then $q^{-1}(\{q(a)\})$ and $q^{-1}(\{q(b)\})$ are disjoint closed subsets of X, and since X is compact Hausdorff it is normal, so we can find disjoint open sets U_a and U_b in X containing $q^{-1}(\{q(a)\})$ and $q^{-1}(\{q(b)\})$, respectively. Now we can construct open sets V_a and V_b in Y such that

$$\begin{aligned} q(a) \in V_a \quad &\text{and} \quad q^{-1}(V_a) \subseteq U_a \\ q(b) \in V_b \quad &\text{and} \quad q^{-1}(V_b) \subseteq U_b \end{aligned}$$

and since U_a and U_b are disjoint from each other, so too are V_a and V_b. We conclude that Y is Hausdorff. ■

Remark. In the above proof it may be tempting to use the fact that $(q \times q)(R) = \Delta$, from which we seem to conclude that Δ is closed in $Y \times Y$ whenever R is closed in $X \times X$; however, this reasoning would be erroneous, because the product of two closed maps need not be a closed map (much unlike the situation for open maps). Thus, in (ii) we needed to use the full strength of the assumption that X was compact and Hausdorff.

6 Quotient manifolds

The analogue of quotient maps for smooth manifolds are smooth submersions, which play much the same role in the study of quotient manifolds as do quotient maps in the study of topological quotient spaces. Facts 13 and 14 and Theorem 7 have direct analogues for smooth submersions. We will start by proving these analogous results because they will be very useful later.

Fact 16. *Let $q : M \to N$ be a smooth submersion. Then q is an open map, and if it's surjective then it's a quotient map.*

Proof. The second statement follows from the first because an open surjective smooth submersion is a quotient map. In order to show that q is an open map, take any open set $W \subseteq M$ and consider the image $q(W) \subseteq N$. For any $q(x) \in q(W)$ use the local submersion theorem (Theorem 1) to select smooth charts φ around x and ψ around $q(x)$ for which $\widehat{q} = \psi \circ f \circ \varphi^{-1}$ is the projection $\mathbb{R}^m \to \mathbb{R}^n$ onto the first n coordinates. Letting $U \subseteq W$ denote the domain of φ and V the domain of ψ, we have

$$q|_U = \psi^{-1} \circ \widehat{q} \circ \varphi : U \xrightarrow{\varphi} \varphi(U) \subseteq \mathbb{R}^m \xrightarrow{\widehat{q}} \psi(V) \subseteq \mathbb{R}^n \xrightarrow{\psi^{-1}} V.$$

Hence $q(U) = \psi^{-1} \circ \widehat{q} \circ \varphi(U) \subseteq q(W)$ is an open neighborhood of y contained in $q(W)$, since the projection $\widehat{q}$ is clearly an open map and ψ and φ are diffeomorphisms. We conclude that $q(W)$ is open and q is an open map. ∎

Fact 17 (Characteristic property of surjective smooth submersions)**.** *Let $q : M \to N$ be a surjective smooth submersions, P any smooth manifold, and $F : N \to P$ any map. Then F is smooth if and only if $F \circ q : M \to P$ is smooth.*

Proof. If F is smooth then $F \circ q$ is smooth by composition. Conversely, suppose that $F \circ q$ is smooth and take any $y \in N$. Since q is surjective we can find some $x \in q^{-1}(y)$, and then apply the local submersion theorem to select smooth charts $\varphi : U \subseteq M \to \mathbb{R}^m$ around x and $\psi : V \subseteq N \to \mathbb{R}^n$ around y (without loss of generality both centered at 0) with respect to which q the local coordinate representation

$$\widehat{q}(x_1, \ldots, x_n, x_{n+1}, \ldots, x_m) = (x_1, \ldots, x_n).$$

Now define $\widehat{\sigma} : \mathbb{R}^n \to \mathbb{R}^m$ by

$$\widehat{\sigma}(x_1, \ldots, x_n) = (x_1, \ldots, x_n, 0, \ldots, 0),$$

and note that $\widehat{q} \circ \widehat{\sigma} = \mathrm{id}_{\mathbb{R}^n}$. Choose an open cube $C \subseteq \psi(V)$ around 0, small enough so that $\widehat{\sigma}(C) \subseteq \varphi(U)$, and define $\sigma : \psi^{-1}(C) \subseteq N \to U \subseteq M$ by

$$\sigma = \varphi^{-1} \circ \widehat{\sigma} \circ \psi : \psi^{-1}(C) \subseteq N \xrightarrow{\psi} C \xrightarrow{\widehat{\sigma}} \varphi(U) \xrightarrow{\varphi^{-1}} U$$

so that inside $\psi^{-1}(C)$ we then have,

$$\begin{aligned} q \circ \sigma &= q \circ (\varphi^{-1} \circ \widehat{\sigma} \circ \psi) \\ &= (\psi^{-1} \circ \widehat{q}) \circ (\widehat{\sigma} \circ \psi) \\ &= \psi^{-1} \circ \mathrm{id}_{\mathbb{R}^n} \circ \psi \\ &= \mathrm{id}|_{\psi^{-1}(C)}. \end{aligned}$$

Therefore

$$F|_{\psi^{-1}(C)} = F|_{\psi^{-1}(C)} \circ \mathrm{id}_{\psi^{-1}(C)} = (F|_{\psi^{-1}(C)} \circ q) \circ \sigma$$

which is smooth as a composition of smooth maps. Having shown that F is smooth in a neighborhood of any point $y \in N$, we conclude that F is smooth. ■

$$\begin{tikzcd} M \arrow[r, "F"] \arrow[d, "q"'] & P \\ N \arrow[ur, "\widetilde{F}"'] & \end{tikzcd}$$

Fact 18 (Descending smoothly to the quotient)**.** *Let $q : M \to N$ be a surjective smooth submersion and $F : M \to P$ any smooth map. Then the following are equivalent:*

(i) There exists a unique smooth map $\widetilde{F} : N \to P$ satisfying $F = \widetilde{F} \circ q$.

(ii) F is constant on the fibers of q; i.e. $q(x_1) = q(x_2) \Rightarrow F(x_1) = F(x_2)$ for every $x_1, x_2 \in M$.

Proof. Note that the map q is a quotient map by Fact 16. Thus by the analogous result for quotient maps, Fact 14, there exists a continuous map $\widetilde{F}$ satisfying $F = \widetilde{F} \circ q$ if and only if condition (ii) holds. Then F is smooth if and only if $\widetilde{F} \circ q$ is smooth, if and only if $\widetilde{F}$ is smooth by the characteristic property of Fact 17. ■

Theorem 8. *If $q_1 : M \to N_1$ and $q_2 : M \to N_2$ are surjective smooth submersions which make the same identifications, then there exists a diffeomorphism $F : N_1 \to N_2$ such that $F \circ q_1 = q_2$.*

Proof. By the analogous result for quotient maps, we know that there is a homeomorphism $F : N_1 \to N_2$ such that $F \circ q_1 = q_2$, so all we need to do is prove that F and F^{-1} are both smooth. But this follows immediately from the characteristic property for surjective smooth submersions because $q_2 = F \circ q_1$ and $q_1 = F^{-1} \circ q_2$ are both smooth. ■

Suppose a topological group G acts continuously on a topological space M, and denote it by $\theta : G \times M \to M$ with $\theta(g, p) = g \cdot p$. The orbit of any $p \in M$ is the subset

$$G \cdot p = \{g \cdot p : g \in G\}$$

and we can define an equivalence relation on M by $p \sim q$ if and only if $q \in G \cdot p$; i.e. if and only if there exists some $g \in G$ such that $q = g \cdot p$. This relation partitions M into orbits of G, and we can take the quotient

$$M/G = \{G \cdot p : p \in M\}$$

which is a topological space equipped with the quotient topology associated with the natural quotient map $q : M \to M/G$ given by $q(x) = G \cdot x$. We call this the ***orbit space*** of the G-action on M.

Fact 19. *Let G be a topological group acting continuously on a topological space M. Then the quotient map $q : M \to M/G$ is an open map.*

Proof. Take any open set $U \subseteq M$. Then $q(U) = \{G \cdot u : u \in U\}$ and therefore

$$\begin{aligned} q^{-1}(q(U)) &= \{x \in M : q(x) \in q(U)\} \\ &= \{x \in M : G \cdot x = G \cdot u \text{ for some } u \in U\} \\ &= \{x \in M : x \in G \cdot u \text{ for some } u \in U\} \\ &= \bigcup_{u \in U} G \cdot u \\ &= \bigcup_{g \in G} g \cdot U \\ &= \bigcup_{g \in G} \theta_g(U) \end{aligned}$$

which is a union of open sets since each θ_g is a homeomorphism of M. Since q is a quotient map, this implies that $q(U)$ is open in M/G and thus q is an open map. ■

We are mostly interested in studying the orbit spaces of smooth Lie group actions on smooth manifolds. Occasionally, because many of the concepts don't require the smooth structure, it suffices to consider continuous actions of topological groups.

Example 7 (Orbit spaces of smooth Lie group actions)**.**

(a) Let G be any group and M any smooth manifold, the trivial action $g \cdot p = p$ foe every $p \in M$ and $g \in G$ has the orbit space $M/G = M$, so the orbit space is trivially a smooth manifold.

(b) $\mathbb{R}^k$ acts on $\mathbb{R}^k \times \mathbb{R}^n$ by translation in the $\mathbb{R}^k$ factor: $v \cdot (x, y) = (v + x, y)$. The orbit of $(x, y) \in \mathbb{R}^k \times \mathbb{R}^n$ is the affine subspace $\mathbb{R}^k \times \{y\}$ parallel to $\mathbb{R}^k$, and hence the orbit space is the smooth manifold

$$(\mathbb{R}^k \times \mathbb{R}^n)/\mathbb{R}^k = \{\mathbb{R}^k \times \{y\} : y \in \mathbb{R}^n\} \simeq \mathbb{R}^n.$$

(c) The circle group S^1 acts on the complex plane $\mathbb{C}$ by multiplication: $z \cdot w = zw$ for every $z \in S^1$ and $w \in \mathbb{C}$. The orbit of $w \in \mathbb{C}$ is the circle of radius $|w|$ centered at the origin, since multiplying by $e^{i\theta} \in S^1$ has the effect of rotating w counterclockwise by an angle of θ.

Let $q : \mathbb{C} \to \mathbb{C}/S^1$ denote the quotient map, then $q(u) = q(v)$ holds if and only if $u = e^{i\theta} v$ for some $\theta \in \mathbb{R}$; i.e. if and only if $|u| = |v|$. Thus we have a continuous map $f : \mathbb{C} \to [0, \infty)$ given by $f(u) = |u|$ making the same identifications as q, and by Theorem 7 we conclude that the orbit space is $\mathbb{C}/S^1$ is homeomorphic to $[0, \infty)$. In particular, the orbit space is not a smooth manifold (but rather a smooth manifold with boundary).

(d) The general linear group $\mathrm{GL}(n, \mathbb{R})$ acts naturally on $\mathbb{R}^n$ via matrix multiplication $(A, x) \mapsto Ax$. As we previously noted, there are exactly two orbits: $\{0\}$ and $\mathbb{R}^n \setminus \{0\}$. Thus the orbit space is

$$X = \mathbb{R}^n / \mathrm{GL}(n, \mathbb{R}) = \{\{0\}, \mathbb{R}^n \setminus \{0\}\}$$

and the open sets in this space are exactly $\varnothing$, X, and $\{\mathbb{R}^n \setminus \{0\}\}$ (the subsets of X whose pre-image under the quotient map are open). So evidently the orbit space is not Hausdorff, because $\{0\}$ cannot be separated from $\mathbb{R}^n \setminus \{0\}$, and thus it's not a manifold.

(e) Restricting the action in (d), the orthogonal group $O(n)$ acts on $\mathbb{R}^n$ via matrix multiplication. Since this action has the effect of rotation, the orbit of any $x \in \mathbb{R}^n$ is the sphere of radius $|x|$ centered at 0 in $\mathbb{R}^n$. As in example (c), the orbit space $\mathbb{R}^n/O(n)$ is homeomorphic to $[0, \infty)$, because the natural quotient map makes the same identifications as the magnitude function $f : \mathbb{R}^n \to [0, \infty), f(x) = |x|$.

(f) Note that if we delete the origin from the previous three examples (c), (d), and (e), we have

$$\begin{aligned} \mathbb{C}^\times/S^1 &\simeq (0, \infty) \\ (\mathbb{R}^n \setminus \{0\})/\operatorname{GL}(n, \mathbb{R}) &\simeq \{\mathbb{R}^n \setminus \{0\}\} \\ (\mathbb{R}^n \setminus \{0\})/O(n) &\simeq (0, \infty) \end{aligned}$$

and evidently all three orbit spaces are now smooth manifolds with this modification.

The modification exhibited in example (f) suggests that we should consider *free* group actions, since in each of the examples (c), (d), and (e) failing to produce smooth orbit spaces, the origin was fixed by every $g \in G$. However, this is not a necessary condition (as example (a) shows), nor is it sufficient, as the following example shows:

Example 8. Let $\mathbb{R}$ act on the torus $\mathbb{T}^2 = S^1 \times S^1$ via an irrational rotation by $\alpha \in \mathbb{R} \setminus \mathbb{Q}$; i.e. for every $t \in \mathbb{R}$ and $(w, z) \in \mathbb{T}^2$ define

$$t \cdot (w, z) = (e^{2\pi i t} w, e^{2\pi i \alpha t} z).$$

It's not difficult to verify that this is a smooth, free action, whose orbits are dense subspaces of $\mathbb{T}^2$. Hence the only open sets in the orbit space $\mathbb{T}^2/\mathbb{R}$ with respect to the quotient topology are $\varnothing$ and $\mathbb{T}^2/\mathbb{R}$ itself; i.e. the orbit space has the trivial topology and is therefore not Hausdorff and also not a manifold.

These examples indicate that we should introduce another restriction if we want to ensure that the orbit space has a smooth manifold structure: we say that a (left) action of a Lie group G on a smooth manifold M is a ***proper action***, or that G ***acts properly*** on M, if the map $\Theta : G \times M \to M \times M$ given by $\Theta(g, p) = (g \cdot p, p)$ is a proper map. Recall that a map $f : X \to Y$ between topological spaces is proper if, for any compact subset $K \subseteq Y$, the pre-image $f^{-1}(K) \subseteq X$ is compact. A similar definition can be made for right Lie group actions.

At this point we should pause for a moment and make a few remarks about the notion of a proper Lie group action. To say that a Lie group G acts properly on M does *not* mean that the map $\theta : G \times M \to M$ defining the action is a proper map; in fact the definition of a proper action is slightly weaker than this. Indeed, suppose that $\theta : G \times M \to M$ is a proper map. We will show that G acts properly on M – i.e. we need to check that the aforementioned map $\Theta : G \times M \to M \times M$ is proper. Let $C \subseteq M \times M$ be compact, and let $\pi_1 : M \times M \to M$ and $\pi_2 : M \times M \to M$ denote the projections onto the first and second factor, respectively. Then $C_1 = \pi_1(C)$

and $C_2 = \pi_2(C)$ are compact, and we have

$$\begin{aligned}\Theta^{-1}(C) &= \{(g,p) : (g \cdot p, p) \in C\} \\ &= \{(g,p) : (\theta(g,p), p) \in C\} \\ &= \{(g,p) : \theta(g,p) \in C_1 \text{ and } p \in C_2\} \\ &= \{(g,p) : (g,p) \in \theta^{-1}(C_1) \text{ and } p \in C_2\} \\ &= \theta^{-1}(\pi_1(C)) \cap (G \times \pi_2(C)).\end{aligned}$$

Note that in the last line $\theta^{-1}(\pi_1(C))$ is compact since we have assumed that θ is a proper map, and $G \times \pi_2(C)$ is closed, hence $\Theta^{-1}(C)$ is compact as a closed subset of a compact set. Thus Θ is a proper map and G acts properly on M.

On the other hand, the converse is *not* true: the group action θ need not be a proper map whenever G acts properly on M. In order to see why, we note the following fact:

Fact 20. *Let G be a Lie group acting continuously on a smooth manifold M. If the group action $\theta : G \times M \to M$ is a proper map, then G is compact.*

Proof. Suppose θ is a proper map. For any $x_0 \in M$, the singleton $\{x_0\} \subseteq M$ is compact, so $\theta^{-1}(x_0) = \{(g,x) : g \cdot x = x_0\} \subseteq G \times M$ is compact. Letting $\pi_1 : G \times M \to G$ denote the projection onto G, the composition

$$\pi_1 \circ i : \theta^{-1}(x_0) \overset{i}{\hookrightarrow} G \times M \xrightarrow{\pi_1} G$$

is surjective: for any $g \in G$ we can find some $x \in M$ for which $g \cdot x = x_0$, namely $x = g^{-1} \cdot x_0$. In other words this pair $(g,x) \in \theta^{-1}(x_0)$ satisfies $(\pi_1 \circ i)(g,x) = g$. Thus $G = \pi_1(i(\theta^{-1}(x_0)))$ is compact because π_1 and i are both continuous. ∎

So in order to construct a counterexample it suffices merely to find a non-compact Lie group G acting properly on a smooth manifold M. For example, $\mathbb{R}^k$ acts properly on $X = \mathbb{R}^k \times \mathbb{R}^n$ by translation in the first k-coordinates, because the pre-image of any compact $K \subseteq X$ under Θ is

$$\Theta^{-1}(K) = \pi_2(K) \cap (\pi_1(K) - (v, 0))$$

where the action translates by $v \in \mathbb{R}^k$, and π_1 and π_2 denote the natural projections onto the $\mathbb{R}^k$ and $\mathbb{R}^n$ factors. Since $\mathbb{R}^k$ is not compact, the group action θ cannot be a proper map. In summary, the properness of the action map is sufficient but not necessary for the group to act properly, and it would be far too strong to define "proper action" in terms of the properness of the action map because such a definition would require the Lie group to be compact.

Fact 21. *If a Lie group G acts continuously and properly on a smooth manifold M then the orbit space M/G is Hausdorff.*

Proof. Since G acts properly on M, the map $\Theta : G \times M \to M \times M$ given by $\Theta(g,p) = (g \cdot p, p)$ is proper. Let $q : M \to M/G$ denote the natural quotient map. As in Fact 15 we consider the set $R = \{(a,b) : q(a) = q(b)\} \subseteq M \times M$. We have

$$\begin{aligned}R &= \{(a,b) : q(a) = q(b)\} \\ &= \{(a,b) : G \cdot a = G \cdot b\} \\ &= \{(a,b) : a = g \cdot b \text{ for some } g \in G\} \\ &= \{(g \cdot b, b) : g \in G, b \in M\} \\ &= \Theta(G \times M).\end{aligned}$$

But Θ is a closed map because proper continuous maps are closed, so R is closed in $M \times M$ and thus M/G is Hausdorff by Fact 15 ∎

Fact 22 (Characterizations of proper actions)**.** *Let G be a Lie group acting continuously on a smooth manifold M. Then the following are equivalent:*

(i) G acts properly on M.

(ii) A sequence (g_i) in G has a convergent sequence whenever there exists a sequence (p_i) in M such that both (p_i) and $(g_i \cdot p_i)$ converge.

(iii) For every compact subset $K \subseteq M$, the set

$$G_K = \{g \in G : (g \cdot K) \cap K \neq \varnothing\} \subseteq G$$

is compact.

Proof. As usual, we let $\Theta : G \times M \to M \times M$ denote the map $\Theta(g,p) = (g \cdot p, p)$.

(i) $\Rightarrow$ (ii). Suppose that G acts properly on M, so that Θ is a proper map. Let (g_i) be a sequence in G, and let (p_i) be a sequence in M such that both (p_i) and $(g_i \cdot p_i)$ converge in M, say $p_i \to p$ and $g_i \cdot p_i \to q$. We need to show that (g_i) has a convergent subsequence.

Since the manifold M is a locally compact Hausdorff space it has a basis consisting of precompact open neighborhoods, hence we can choose two precompact open neighborhoods U and V around p and q, respectively. Thus, when i is large enough, $\Theta(g_i, p_i) = (g_i \cdot p_i, p_i)$ lies in the compact set $\overline{V} \times \overline{U} \subseteq M \times M$. So the pair (g_i, p_i) lies in the compact pre-image $\Theta^{-1}(\overline{V} \times \overline{U})$, hence this sequence of pairs has a convergent subsequence, and so (g_i) also has a convergent subsequence.

(ii) $\Rightarrow$ (iii). Let $K \subseteq M$ be compact. Aiming to show that G_K is sequentially compact, take any sequence (g_i) in G_K. Then for every i, the intersection $(g_i \cdot K) \cap K$ is nonempty, and we can select a point $p_i \in (g_i \cdot K) \cap K$; in particular, $p_i \in K$ and $g_i^{-1} \cdot p_i \in K$ for every i. Since K is compact, (p_i) has a convergent subsequence in K and without loss of generality we assume that (p_i) converges. Similarly, $(g_i^{-1} \cdot p_i)$ has a convergent subsequence, say $(g_{i_j}^{-1} \cdot p_{i_j}) \subseteq (g_i^{-1} \cdot p_i)$ in K. Thus $(g_{i_j}^{-1})$ is a sequence in G such that both (p_{i_j}) and $(g_{i_j}^{-1} \cdot p_{i_j})$ converge in M, and by condition (ii) this means that $(g_{i_j}^{-1})$ has a convergent subsequence. This also means that (g_{i_j}) has a convergent subsequence, which yields a convergent subsequence of the original sequence (g_i). We conclude that G_K is sequentially compact, as desired.

(iii) $\Rightarrow$ (i). Take any compact $L \subseteq M \times M$. We need to show that $\Theta^{-1}(L) \subseteq G \times M$ is compact. Let $\pi_1, \pi_2 : M \times M \to M$ denote the projections onto the first and second factors, respectively, and set $K = \pi_1(L) \cup \pi_2(L) \subseteq M$. Note that K is compact as the union of two compact sets. We have

$$\begin{aligned}\Theta^{-1}(L) &= \{(g,p) \in G \times M : (g \cdot p, p) \in L\} \\ &= (g,p) : g \cdot p \in \pi_1(L) \text{ and } p \in \pi_2(L)\}\end{aligned}$$

and note that $g \cdot p \in \pi_1(L)$ implies that $p \in g^{-1} \cdot \pi_1(L)$. Thus, $\theta^{-1}(L)$ consists of precisely those pairs (g,p) such that $p \in (g^{-1} \cdot \pi_1(L)) \cap \pi_2(L)$. But this latter set is contained inside $(g^{-1} \cdot K) \cap K$ because K contains both $\pi_1(L)$ and $\pi_2(L)$ by definition. Therefore

$$\Theta^{-1}(L) \subseteq \{(g,p) : p \in (g^{-1} \cdot K) \cap K\} \subseteq G_K \times K$$

and G_K is compact by assumption, so $\Theta^{-1}(L)$ is compact as a closed subset of a compact set. We conclude that Θ is a proper map and G acts properly on M. ∎

Heuristically, the set G_K defined in condition (iii) of Fact 22 consists of those group elements which move K only slightly, so that the new set still intersects K. Thus, to say that G_K is compact means intuitively that almost every group element moves K far away from itself.

Corollary 4. *Every continuous action by a compact Lie group on a smooth manifold is proper.*

Proof. In this case, condition (ii) in Fact 22 is trivially satisfied because every sequence in the Lie group has a convergent subsequence by compactness. ∎

With this corollary we have slightly refined an implication we previously established with Fact 20. In summary we have completed the following sequence of implications for a Lie group G acting continuously on a smooth manifold M: [the G-action θ is a proper map] $\Rightarrow$ [G is compact] $\Rightarrow$ [G acts properly on M].

Fact 23 (Orbits of proper actions)**.** *Suppose a Lie group G acts smoothly and properly on a smooth manifold M. Then for any $p \in M$ the orbit map $\theta^{(p)}$ is proper, and consequently:*

(i) Orbits $G \cdot p = \theta^{(p)}(G)$ are closed in M.

(ii) Stabilizers $G_p = \left(\theta^{(p)}\right)^{-1}(p)$ are compact subgroups of G.

(iii) If $G_p = \{e\}$ then $\theta^{(p)}$ is a smooth embedding and $G \cdot p$ is an embedded submanifold.

Proof. We fix some $p \in G$ and show first of all that $\theta^{(p)}$ is a proper map. Taking any compact $K \subseteq M$, the pre-image $\left(\theta^{(p)}\right)^{-1}(K)$ is closed in G by continuity, and moreover

$$\begin{aligned}\left(\theta^{(p)}\right)^{-1}(K) &= \{g \in G : g \cdot p \in K\} \\ &= \{g \in G : p \in g^{-1} \cdot K\} \\ &\subseteq G_{K \cup \{p\}}\end{aligned}$$

because if $g \in \left(\theta^{(p)}\right)^{-1}(K)$ then $p \in g^{-1} \cdot K$ certainly implies $p \in g^{-1} \cdot (K \cup \{p\})$, hence $p \in (g^{-1} \cdot K \cup \{p\}) \cap (K \cup \{p\})$ and thus $g \in G_{K \cup \{p\}}$. Since G acts properly, this latter set is compact by Fact 22, and $\left(\theta^{(p)}\right)^{-1}(K)$ is compact as a closed subset of a compact set. Consequently:

(i) $\theta^{(p)}$ is a closed map because proper continuous maps are closed, hence the orbit $G \cdot p$ is closed as the image of G under a closed map.

(ii) Each stabilizer G_p is compact as the pre-image of the compact set $\{p\}$ under the proper map $\theta^{(p)}$, and they're also subgroups of G for algebraic reasons.

(iii) If $G_p = \{e\}$ then by Fact 11 the orbit map $\theta^{(p)}$ is an injective smooth immersion, but it's also a closed map, so it's a smooth embedding. Hence $G \cdot p = \theta^{(p)}$ is an embedded submanifold of M.

∎

Finally, we can state a very important theorem which establishes sufficient conditions for the orbit space M/G of a smooth Lie group action to be a smooth manifold; although we will forego the proof because it involves the machinery of foliations and integral manifolds of geometric distributions.

Theorem 9 (Quotient manifold theorem)**.** *Let G be a Lie group acting smoothly, freely, and properly on a smooth manifold M. Then:*

- *The orbit space M/G is a topological manifold with dimension* $\dim M/G = \dim M - \dim G$.
- *M/G admits a unique smooth structure for which the natural quotient map $q : M \to M/G$ is a smooth submersion.*

Next we will use the quotient manifold theorem to construct and characterize an important class of manifolds: those equipped with transitive Lie group actions.

7 Homogeneous spaces

One of the most important aspect of Lie groups is that any point can be mapped to any other via a diffeomorphism; namely $L_{gh^{-1}}(h) = gh^{-1}h = g$ for any $g, h \in G$. In other words, every Lie group is endowed with a transitive smooth action by a Lie group (just acting on itself via left multiplication). In general, a smooth manifold with this property, that there is a Lie group G acting smoothly and transitively upon it, is called a ***homogeneous space*** or a ***homogeneous G-space***. Of course, we could relax several components of this definition and say that a homogeneous space is a topological space equipped with a continuous transitive action by a topological group, but for the purposes of this note we will just focus on the setting of smooth manifolds and Lie groups.

Our goal in this section is to use the quotient manifold theorem to prove two fundamental results about homogeneous spaces: a construction theorem and a characterization theorem. First we start with some important examples of homogeneous spaces.

Example 9 (Homogeneous spaces)**.**

(a) The natural action of $O(n)$ on $S^{n-1} \subseteq \mathbb{R}^n$ via matrix multiplication is transitive because any unit vector can be mapped to any other by an appropriate combination of rotations. Hence S^{n-1} is a homogeneous $O(n)$-space.

(b) The action of $O(n)$ on S^{n-1} restricts to an action of $\mathrm{SO}(n)$ on S^{n-1}. For $n = 1$ this action is trivial because $\mathrm{SO}(1)$ is the trivial group, and for $n > 1$ the action is transitive by the same reasoning as above. Thus S^{n-1} is a homogeneous $\mathrm{SO}(n)$-space for $n \geq 2$.

(c) $\mathrm{SL}(2, \mathbb{R})$ acts smoothly and transitively on the upper-half complex plane $U = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ by the formula

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot z = \frac{az + b}{cz + d}.$$

For any $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}(2, \mathbb{R})$, the complex-analytic diffeomorphism $U \to U$ given by $z \mapsto (az + b)/(cz + d)$ is called a Mobius transformation. Hence the upper-half plane U is a homogeneous $\mathrm{SL}(2, \mathbb{R})$-space.

(d) For $n \geq 1$, the natural action of $\mathrm{GL}(n, \mathbb{C})$ on $\mathbb{C}^n$ restricts to a smooth action of $U(n)$ on $S^{2n-1} \subseteq \mathbb{C}^n$. This action is transitive for $n \geq 1$. Similarly, the action of $\mathrm{GL}(n, \mathbb{C})$ restricts to a smooth action of $\mathrm{SU}(n)$ on S^{2n-1}, and this action is transitive for $n \geq 2$.

Let G be a Lie group and $H \subseteq G$ any Lie subgroup. Recall that a ***left coset*** of H is a subset of G of the form

$$gH = \{gh : h \in H\}$$

for some $g \in G$. The left cosets of H partition G, as they correspond to equivalence classes of the equivalence relation $a \sim b$ if and only if $aH = bH$; i.e. $a \sim b$ if and only if $a^{-1}b \in H$. The resulting quotient space is

$$G/H = \{gH : g \in G\},$$

equipped with the quotient topology. We call this space the ***left coset space*** of G modulo H. Notice that, if we let H act on G by *right* multiplication,

$$H \times G \to G, (h, g) \mapsto gh,$$

then the orbits are the *left* cosets $H \cdot g = gH$. Thus the left coset space G/H is exactly the orbit space determined by the action of right multiplication of H on G.

Theorem 10 (Homogeneous space construction theorem)**.** *Let G be a Lie group and $H \subseteq G$ any closed Lie subgroup. Then:*

(i) G/H is a topological manifold with dimension $\dim G/H = \dim G - dimH$.

(ii) G/H has a unique smooth structure such that the natural quotient map $q : G \to G/H$ is a smooth submersion.

(iii) G/H is a homogeneous G-space with respect to the left G-action $a \cdot (bH) = abH$.

Proof. Throughout the proof, we consider the right multiplication action of H on G (so that the orbit space is G/H). We start with the following straightforward observations:

- $H \subseteq G$ is an embedded Lie subgroup of G by Theorem 5, so the inclusion $H \hookrightarrow G$ is a smooth embedding. Thus the action of H on G is smooth as the restriction of the smooth multiplication map in G.
- H acts freely on G because $g = h \cdot g = gh$ implies that $h = e$.
- H acts properly on M. Indeed, let (h_i) be any sequence in H and suppose we can find a sequence (g_i) in G such that both (g_i) and $(g_i h_i)$ converge in G. Then (g_i^{-1}) also converges by continuity of the inversion map, and

$$h_i = g_i^{-1}(g_i h_i) = m(g_i^{-1}, g_i h_i)$$

so (h_i) converges in G by continuity of the multiplication map m. Since $H \subseteq G$ is closed and $(h_i) \subseteq H$ we conclude that (h_i) converges in H. Thus the action is proper by Fact 22.

Since H acts smoothly, freely and properly on G to produce the orbit space G/H, the quotient manifold theorem shows that G/H is a topological manifold with dimension $\dim G/H = \dim G - \dim H$, with a unique smooth structure making the natural quotient map a smooth submersion. All that remains is to check that G/H is a homogeneous G-space; i.e. that the action $\theta : G \times G/H \to G/H$ is smooth and transitive.

Since q is a surjective smooth submersion, so too is $\mathrm{id}_G \times q : G \times G \to G \times G/H$, and it's easy to check that $q \circ m$ is constant on the fibers of $\mathrm{id}_G \times q$. Thus by Fact 18 there is a unique smooth map $G \times G/H \to G/H$ completing the diagram; in fact it must be the action θ because the latter satisfies $\theta \circ (\mathrm{id}_G \times q) = q \circ m$. We conclude that θ is well-defined and smooth, and it's also transitive because for any $a, b \in G$ we have $(ba^{-1}) \cdot aH = bH$. ∎

Due to the closed subgroup theorem (Theorem 5), we can actually remove the assumption that $H \subseteq G$ be a closed *Lie* subgroup; in fact it suffices to assume that H is any closed subgroup. This significantly expands our class of known homogeneous spaces.

In summary, the construction theorem tells us that we can construct examples of homogeneous spaces by taking left coset spaces of Lie groups modulo closed subgroups. The following theorem actually shows that *every* homogeneous space has the form given by the construction theorem, i.e. every homogeneous space is a quotient of a Lie group by some closed subgroup.

Theorem 11 (Homogeneous space characterization theorem)**.** *Let G be a Lie group and let M be a homogeneous G-space. For any $p \in M$, the stabilizer G_p is a closed subgroup of G, and the map $F : G/G_p \to M$ given by $F(gG_p) = g \cdot p$ is an equivariant diffeomorphism.*

Proof. The stabilizer is a subgroup for algebraic reasons, and it's closed by continuity of the orbit map $\theta^{(p)} : G \to M$. Hence by Theorem 10 we know that G/G_p is a topological manifold admitting a unique smooth structure such that the natural quotient map $q : G \to G/H$ is a smooth submersion. Note that $\theta^{(p)}$ is constant on the fibers of q because for every $g_1, g_2 \in G$, we have

$$\begin{aligned} q(g_1) = q(g_2) &\Rightarrow g_1 G_p = g_2 G_p \\ &\Rightarrow g_2^{-1} g_1 = h \in G_p \\ &\Rightarrow \theta^{(p)}(g_1) = \theta^{(p)}(g_2 h) \end{aligned}$$

but $\theta^{(p)}(g_2 h) = (g_2 h) \cdot p = g_2 \cdot (h \cdot p) = g_2 \cdot p = \theta^{(p)}(g_2)$ since h stabilizes p. Hence $\theta^{(p)}(g_1) = \theta^{(p)}(g_2)$. Then by Fact 18 $\theta^{(p)}$ descends to the quotient, so there exists a smooth map $F : G/G_p \to M$ satisfying $F \circ q = \theta^{(p)}$; that is, $F(gG_p) = g \cdot p$ for every $g \in G$.

In fact, $\theta^{(p)}$ is surjective because the G-action is transitive, and it has constant rank by Fact 11, so it's also a smooth submersion. It's easy to check that q is constant on the fibers of $\theta^{(p)}$ (so that q and $\theta^{(p)}$ make the same identifications), thus F is a diffeomorphism by Theorem 8.

Finally, F is equivariant with respect to the G-actions on G/G_p and M because for every $g, h \in G$ we have

$$\begin{aligned} F(g \cdot (hG_p)) &= F(ghG_p) \\ &= (gh) \cdot p \\ &= g \cdot (h \cdot p) \\ &= g \cdot F(hG_p) \end{aligned}$$

∎

Thus the characterization theorem says that every homogeneous space arises as the left coset space of a Lie group modulo some closed subgroup. As a result, the study of homogeneous spaces can be reduced to the problem of understanding quotients of Lie groups by closed subgroups: if M is a smooth manifold and G is a Lie group acting transitively on M, then $M \simeq G/H$ for some closed subgroup $H \subseteq G$, and we can analyze M using all of the available machinery for analyzing quotient manifolds and orbit spaces. Due to this characterization, in the literature on homogeneous spaces, authors will simply write "let G/H be a homogeneous space". Let's revisit the previous examples of homogeneous spaces to see what the characterization theorem tells us about them.

Example 10 (Homogeneous spaces revisited)**.**

(a) Consider the natural action of $O(n)$ on $S^{n-1} \subseteq \mathbb{R}^n$ for $n \geq 1$. Fix the north pole $N = (0, \ldots, 0, 1)$ as our base point. The stabilizer of N in $O(n)$ is the subgroup of orthogonal transformations of $\mathbb{R}^n$ that fix the last coordinate, i.e. $O(n-1)$. In the case $n = 3$, think of holding a ball between two fingers and spinning it around the z-axis: these transformations are precisely the rotations of S^1. Thus the characterization theorem tells us that S^{n-1} is diffeomorphic to the quotient manifold $O(n)/O(n-1)$ for $n \geq 1$.

(b) Consider the natural action of $\mathrm{SO}(n)$ on $S^{n-1} \subseteq \mathbb{R}^n$ for $n \geq 2$. By the same reasoning as before, the stabilizer subgroup of the north pole $N \in S^{n-1}$ is $\mathrm{SO}(n-1)$, so S^{n-1} is diffeomorphic to the quotient manifold $\mathrm{SO}(n)/\mathrm{SO}(n-1)$ for $n \geq 2$.

(c) Consider the action of $\mathrm{SL}(2, \mathbb{R})$ on the upper-half plane $U \subseteq \mathbb{C}$ via Mobius transformations. Fix $i \in U$ as our base point. Which matrices $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}(2, \mathbb{R})$ stabilize i? We can solve the equation

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot i = i \iff \frac{ai + b}{ci + d} = i$$

to find that this holds if and only if $a = d$ and $b = -c$, which in turn means that $A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ with $1 = \det A = a^2 + b^2$. This is precisely the subgroup $\mathrm{SO}(2)$ of 2×2 special orthogonal matrices. So the characterization theorem tells us that we have a diffeomorphism $U \simeq \mathrm{SL}(2, \mathbb{R})/\mathrm{SO}(2)$.

The homogeneous space characterization theorem can also be used to construct new examples of smooth manifolds: we can use it to put a smooth manifold structure on *sets* that admit transitive Lie group actions.

Theorem 12. *Let X be a set and let G be a Lie group acting transitively on X such that the stabilizer G_p is a closed subgroup of G for some $p \in X$. Then X has a unique smooth manifold structure for which the G-action is smooth, and* $\dim X = \dim G - \dim G_p$.

Proof. By the characterization theorem, G/G_p is a homogeneous G-space with $\dim G/G_p = \dim G - \dim G_p$. The natural quotient map $q : G \to G/G_p$ is a surjective smooth submersion, and (just as in the proof of Theorem 11) $\theta^{(p)}$ is a surjective map making the same identifications as q. Hence we get an equivariant bijection $F : G/G_p \to X$ given by $F(gG_p) = g \cdot p$. We define a topology and smooth structure on X by demanding that F be a diffeomorphism; then, with respect to this structure, the action of G on X is smooth because it can be expressed as

$$\begin{aligned} g \cdot x &= g \cdot F(F^{-1}(x)) \\ &= F(g \cdot F^{-1}(x)) \quad \text{(since } F \text{ is equivariant)} \end{aligned}$$

which is a composition of smooth maps.

It remains to show that this smooth manifold structure is unique. Let $\widetilde{X}$ denote X equipped with any other smooth manifold structure for which the G-action on X is smooth. Then $\widetilde{X}$ and X are both homogeneous G-spaces, so by the characterization theorem we have diffeomorphisms $X \simeq G/G_p \simeq \widetilde{X}$. ■

Example 11 (Grassmannian manifolds)**.** Let V be an n-dimensional real vector space. For any $0 \leq k \leq n$, let $G_k(V)$ denote the set of all k-dimensional linear subspaces of V. We will show that $G_k(V)$ can be given the structure of a smooth manifold of dimension $k(n-k)$, called a ***Grassmannian manifold***. Note that this construction is a direct generalization of the real projective space because $G_1(\mathbb{R}^{n+1}) = \mathbb{R}\mathrm{P}^n$.

We define a transitive action of $\mathrm{GL}(n, \mathbb{R})$ on $G_k(\mathbb{R}^n)$ as follows: given any subspace A of $\mathbb{R}^n$, let $T \in \mathrm{GL}(n, \mathbb{R})$ act on A by setting $T(A) = \{T(a) : a \in A\}$; i.e. $T(A)$ is the image of the restriction of T to the subspace A. Note that a basis for $T(A)$ can be obtained by applying T to any basis for A. We fix the base point $\mathbb{R}^k \times \{0\} \subseteq \mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k}$ and note that the stabilizer group of $\mathbb{R}^k \times \{0\} \in G_k(V)$ is

$$H = \left\{ \begin{bmatrix} A & B \\ 0 & D \end{bmatrix} : A \in \mathrm{GL}(k, \mathbb{R}), B \in M(k \times (n-k), \mathbb{R}), D \in \mathrm{GL}(n-k, \mathbb{R}) \right\}.$$

Here's the reason: first of all A and D must be invertible so that the block matrix is invertible (recall the determinant formula for upper triangular block matrices). If the block matrix is going to fix the subspace $\mathbb{R}^k \times \{0\}$ then it needs to send any vector of the form $v = (v_1, \ldots, v_k, 0, \ldots, 0)$ to another vector of the same form. Thus B can be any $k \times (n-k)$ matrix because it acts on the last $n-k$ zero entries of v, and the lower left $(n-k) \times k$ block must be zero in order to preserve the last $n-k$ zeros of v.

Moreover, note that H is a closed subgroup of $\mathrm{GL}(n, \mathbb{R})$: if $\begin{bmatrix} A_i & B_i \\ 0 & D_i \end{bmatrix}$ is a sequence in H such that

$$\begin{bmatrix} A_i & B_i \\ 0 & D_i \end{bmatrix} \to \begin{bmatrix} A & B \\ 0 & D \end{bmatrix} \in \mathrm{GL}(n, \mathbb{R})$$

then

$$\det \begin{bmatrix} A & B \\ 0 & D \end{bmatrix} = \det(A)\det(D) \neq 0$$

which implies that $\det(A) \neq 0$ and $\det(D) \neq 0$, hence $A \in \mathrm{GL}(k, \mathbb{R})$ and $D \in \mathrm{GL}(n-k, \mathbb{R})$, and the limit point lies in H.

Therefore, by Theorem 12 there is a unique smooth structure on $G_k(\mathbb{R}^n)$ for which the $\mathrm{GL}(n, \mathbb{R})$-action is smooth, and we have an equivariant diffeomorphism $G_k(\mathbb{R}^n) \simeq \mathrm{GL}(n, \mathbb{R})/H$. Finally, the dimension of the submanifold H is

$$\dim H = k(n-k) + k^2 + (n-k)^2 = n^2 - k(n-k),$$

so the dimension of $G_k(\mathbb{R}^n)$ is

$$\dim G_k(\mathbb{R}^n) = \dim \mathrm{GL}(n,\mathbb{R}) - \dim H = n^2 - (n^2 - k(n-k)) = k(n-k).$$

Summary (Constructing and characterizing homogeneous spaces)**.** Suppose we have a Lie group G acting smoothly and transitively on a smooth manifold M, and we want to characterize M as an orbit space of G. Then we can proceed as follows:

1. Fix any base point $p \in M$.
2. Show that the stabilizer G_p is a closed subgroup of G.
3. Then the characterization theorem tells us that M is equivariantly diffeomorphic to G/G_p.

Suppose we have a set X and a Lie group G acting transitively on X, and we want to put a smooth manifold structure on X. Then we can proceed as follows:

1. Fix any base point $x \in X$.
2. Show that the stabilizer G_x is a closed subgroup of G.
3. Then Theorem 12 tells us that X is a smooth manifold, equivariantly diffeomorphic to G/G_x.

8 Connectedness in matrix groups

In this section we will use the matrix group characterizations we established in Example 10 to analyze the connectedness and connected components of several important matrix groups.

Fact 24. *Let G be a topological group acting continuously, freely, and properly on a topological space M. If G and M/G are connected, then M is connected.*

Proof. Suppose for contradiction that G and M/G are connected, but M is not connected. Thus we can find disjoint nonempty open subsets $U, V \subseteq M$ such that $U \cup V = M$. For each $p \in M$, the orbit is $G \cdot p = \theta^{(p)}(G) \subseteq M$ which is connected as the image of a connected set under a continuous map, hence each orbit is completely contained in either U or V. The natural quotient map $q : M \to M/G$ is a smooth submersion, hence also an open map, so $q(U)$ and $q(V)$ are open subsets of M/G. Also, $q(U)$ and $q(V)$ are disjoint: if $u \in U$ and $v \in V$ are such that $q(u) = q(v) \in q(U) \cap q(V)$, then $g \cdot u = v$ for some $g \in G$, which implies that u and v lie in the same orbit, contradicting the fact that each orbit is fully contained in either U or V. Therefore $M/G = q(M) = q(U \cup V) = q(U) \cup q(V)$ is a separation of M/G, contradicting the assumption that M/G is connected. ∎

In the proof of the following fact we use the characterizations of matrix groups as homogeneous spaces, as we previously described in Example 10.

Fact 25 (Connectedness of orthogonal and unitary groups)**.** *For every $n \geq 1$, the Lie groups $\mathrm{SO}(n)$, $U(n)$, and $\mathrm{SU}(n)$ are connected. Moreover, $O(n)$ has exactly two connected components, the identity component being $\mathrm{SO}(n)$.*

Proof.

- $\mathrm{SO}(n)$ is connected. For $n = 1$, $\mathrm{SO}(1)$ is the trivial group, hence connected. Suppose that $\mathrm{SO}(n-1)$ is connected for some $n \geq 2$. Then $\mathrm{SO}(n)/\mathrm{SO}(n-1) \simeq S^{n-1}$ is connected, which implies that $\mathrm{SO}(n)$ is connected by Fact 24. Thus each $\mathrm{SO}(n)$ is connected by induction.
- $U(n)$ and $\mathrm{SU}(n)$ are connected. This follows from a similar argument as above, since $U(1) \simeq S^1$ is connected and for every $n \geq 2$ we have $U(n)/U(n-1) \simeq S^{2n-1}$ and $\mathrm{SU}(n)/\mathrm{SU}(n-1) \simeq S^{2n-1}$.
- $O(n)$ has exactly two components. We can write the orthogonal group as a union $O(n) = O^+(n) \cup O^-(n)$ of those orthogonal matrices with positive and negative determinant (i.e. ± 1). It suffices to check that these two subsets are connected. $O^+(n) = \mathrm{SO}(n)$ is connected by the above, and $O^-(n)$ is connected because it's diffeomorphic to $\mathrm{SO}(n)$ via left translation by any reflection in $O(n)$.

■

Fact 26 (Connectedness of the general linear group)**.** *The Lie group* $\mathrm{GL}(n, \mathbb{R})$ *has exactly two connected components: the open subgroups* $\mathrm{GL}^+(n, \mathbb{R})$ *and* $\mathrm{GL}^-(n, \mathbb{R})$ *consisting of those matrices with positive and negative determinant, respectively.*

Proof. The separation $\mathbb{R}^\times = \mathbb{R}^+ \cup \mathbb{R}^-$ yields a separation

$$\begin{aligned} \mathrm{GL}(n, \mathbb{R}) &= (\det)^{-1}(\mathbb{R}^\times) \\ &= (\det)^{-1}(\mathbb{R}^+) \cup (\det)^{-1}(\mathbb{R}^-) \\ &= \mathrm{GL}^+(n, \mathbb{R}) \cup \mathrm{GL}^-(n, \mathbb{R}) \end{aligned}$$

so it suffices to show that these latter two subsets are (path) connected. We will prove this by appealing to the decomposition of any invertible matrix into a product of elementary matrices. For any matrix $A \in \mathrm{GL}^+(n, \mathbb{R})$, we can write A as a product of elementary matrices

$$A = E_1 E_2 \cdots E_k$$

where each E_i has positive determinant. By definition, each matrix E_i is the $n \times n$ identity matrix with exactly one entry changed, say the (p_i, q_i) entry. Thus we can write it as

$$E_i = I + D(p_i, q_i)$$

where each matrix $D(p_i, q_i)$ is a matrix of zeros except for the (p_i, q_i) entry. The statement that each elementary matrix has positive determinant essentially amounts to saying that no row swaps are required when row-reducing A. In particular, note that we have a smooth path $\gamma_i : [0, 1] \to \mathrm{GL}^+(n, \mathbb{R})$ from the identity matrix I to E_i,

$$\gamma_i(t) = I + tD(p_i, q_i)$$

Thus, we can combine these paths to get a smooth path $\gamma(t) = \gamma_1(t) \cdots \gamma_k(t)$ from I to A, and we conclude that $\mathrm{GL}^+(n, \mathbb{R})$ is connected. As for the matrices with negative determinant, any matrix B with $\det B < 0$ yields a diffeomorphism by left-multiplication $L_B : \mathrm{GL}^+(n, \mathbb{R}) \to \mathrm{GL}^-(n, \mathbb{R})$ so $\mathrm{GL}^-(n, \mathbb{R})$ is connected as well. This completes the proof. ■

9 Isomorphism theorem for Lie groups

Another application of the quotient manifold theorem is to extend the first isomorphishm theorem from group theory to the setting of Lie groups.

Fact 27. *Let G be a Lie group and $K \subseteq G$ a closed normal subgroup. Then the quotient group G/K is a Lie group and the natural quotient map $q : G \to G/K$ is a surjective Lie group homomorphism with kernel K.*

Proof. We already know from group theory that the quotient G/K is a group and $q : G \to G/K$ is a surjective group homomorphism. The remainder of the statement (G/K is a Lie group, and q is smooth) follows directly from the construction theorem (Theorem 10). ■

Theorem 13 (First isomorphism theorem for Lie groups)**.** *Let F be a Lie group homomorphism. Then:*

(i) $\ker F$ *is a closed normal Lie subgroup of* G*.*

(ii) $\operatorname{im} F$ *has a unique smooth manifold structure making it into a Lie subgroup of* H*.*

(iii) F *descends to a Lie group isomorphism* $\widetilde{F} : G/\ker F \to \operatorname{im} F$*.*

In particular, if F is surjective then $\widetilde{F}$ yields an isomorphism of Lie groups $G/\ker F \simeq H$.

Proof. The kernel $\ker F$ is a normal subgroup of G for algebraic reasons, and it's closed as the pre-image of $e \in G$ under the continuous map F. Hence it's a Lie subgroup by the closed subgroup theorem (Theorem 5), and $G/\ker F$ is a Lie group (as in Fact 27). The first isomorphism theorem for groups gives us an injective group homomorphism $\widetilde{F} : G/\ker F \to H$ with image $\operatorname{im} \widetilde{F} = \operatorname{im} F$. The canonical projection $q : G \to G/\ker F$ is a surjective Lie group homomorphism, hence it has constant rank by Fact 4, and so it's also a smooth submersion. Since $F = \widetilde{F} \circ q$ is smooth, we conclude by means of the characteristic property for smooth submersions that $\widetilde{F}$ is smooth.

As a result, $\widetilde{F}$ is an injective Lie group homomorphism, hence it has constant rank, and so it's also a smooth immersion. Thus $\operatorname{im} F = \operatorname{im} \widetilde{F}$ is endowed with a unique smooth structure making it into an immersed submanifold (and hence Lie subgroup) of H. Moreover, we recall that a Lie group homomorphism is bijective if and only if it's a diffeomorphism, so $\widetilde{F}$ is a Lie group isomorphism. ■

10 References

In this note we mostly followed John Lee's *Intro to Smooth Manifolds* (pp. 150-170 and 540-560), filling in details to several exercises and problems along the way.