Conservation laws for Yang-Mills Lagrangians

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1. Introduction

Broadly speaking, gauge theory is the study of connections on principal bundles. Given a Lagrangian on a principal bundle, one obtains an Euler-Lagrange operator, then one can define special connections associated with the variational problem (critical points of the action functional). Starting with this idea, there are a wide variety of directions one could pursue. See for instance [AJ78, DK97, CVB03, H17]. Let us briefly outline two distinct but interrelated directions of gauge theory:

1. *Field theory*: gauge theory provides a common mathematical framework for describing various field theories arising in physics. For example, electromagnetism, gravitation, quantum field theory, etc. In this context, fields are sections of configuration bundles associated to a given principal bundle, the potentials correspond to connections, and the field strength is the curvature of the connection. Conservation

laws are then understood as consequences of Noether's theorem, arising from gauge symmetries of the action functional. The aim of gauge-theoretic field theory is to identify natural geometric structures (bundles, connections, metrics, Lagrangians) whose critical points reproduce the fundamental equations of physics.

2. *Smooth invariants*: Distinguishing smooth structures on 4-manifolds is difficult. Topological invariants like Betti numbers or Euler characteristic are not sufficient. In order to obtain an interesting invariant of the smooth structure, the general idea is to fix some additional geometric structure, for example a principal bundle with connection, then write down a non-linear PDE from that structure, and then study the topology of the moduli space of solutions. If one is lucky, this does not depend on the additional choices made, only on the smooth structure. There are many examples of this procedure, including Donaldson's Yang-Mills invariants, Seiberg-Witten theory, Bauer-Furuta theory, etc.

In this note, we will focus on pursuing the first direction. In addition to being interesting and useful in its own right, it turns out that the first direction is useful for the second. Although we will focus on the mathematics, it is instructive and interesting to relate the physical motivation associated with the mathematical formalism, so we will frequently do so. For example, in the U(1) case the Yang-Mills equations reduce to the vacuum Maxwell's equation governing electromagnetic fields. Our primary references are [DK97, NS96, T94, CVB03, H17]. We will fill in certain details along the way that are often left out of the literature, and summarize some of the main themes.

1.1. Preliminaries. In this section we introduce our notation and conventions, and the basic objects we will be working with.

Throughout this note we will consider G-principal bundles $\pi_P : P \to X$ with structure group a real Lie group G, $\dim G > 0$. By definition, $P \to X$ is a fiber bundle equipped with a free transitive right G-action on P, and equipped with a bundle atlas $\{(U_\alpha, \psi_\alpha)\}$ whose trivialization morphisms

$$\psi_\alpha^P : \pi_P^{-1}(U_\alpha) \to U_\alpha \times G$$

are G-equivariant in the sense that

$$(\mathrm{pr}_2 \circ \psi_\alpha^P)(pg) = (\mathrm{pr}_2 \circ \psi_\alpha^P)(p)g$$

for every $g \in G$ and $p \in \pi_P^{-1}(U_\alpha)$. Owing to to this property, every trivialization morphism ψ_α determines a unique local section $z_\alpha : U_\alpha \to P$ such that

$$\mathrm{pr}_2 \circ \psi_\alpha \circ z_\alpha = 1,$$

where 1 is the unit element of G. The transformation rules for z_α are given by

$$z_\beta(x) = z_\alpha(x)\rho_{\alpha\beta}(x), \quad x \in U_\alpha \cap U_\beta, \tag{1.1}$$

where $\rho_{\alpha\beta}(x, g) = \rho_{\alpha\beta}(x)g$ are the transition functions for the bundle atlas of P. Conversely, the family $\{(U_\alpha, z_\alpha)\}$ of local sections of P obeying (1.1) uniquely determines a bundle atlas for P. Note that there is a pull-back operation on the principal bundle structure: the pull-back f^*P of a principal bundle is also a principal bundle within the same structure group.

Let G be a Lie group. For $g \in G$ we let L_g and R_g denote the left and right multiplication automorphisms. We let $\mathfrak{g}_l$ denote the left Lie algebra of G, consisting of left-invariant vector fields on G. We will let $\{\epsilon_m\}$ denote a basis for $\mathfrak{g}_l$, and let c^k_{mn} denote the structure constants, so that

$$[\epsilon_m, \epsilon_n] = c^k_{mn}\epsilon_k.$$

As usual, there is a natural isomorphism of $\mathfrak{g}_l$ with the tangent space T_1G given by assocating any $v \in T_1G$ with the left-invariant vector field $g \mapsto L_g v$ on G. Of course, one may also consider the right Lie algebra $\mathfrak{g}_r$.

The left action L_g of a Lie group G on itself defines its adjoint representation $g \mapsto \mathrm{Ad}_g$ acting on the right Lie algebra $\mathfrak{g}_r$, and its identity representation acting on the left Lie algebra $\mathfrak{g}_l$. Consequently we have the adjoint representation

$$\begin{aligned} \mathrm{ad}_{\epsilon'} &: \mathfrak{g}_r \to \mathfrak{g}_r \\ \mathrm{ad}_{\epsilon'}(\epsilon) &= [\epsilon', \epsilon] \end{aligned}$$

of the right Lie algebra $\mathfrak{g}_r$ on itself.

Any left action $G \times Z \to Z$ of a Lie group G on a manifold Z yields a homomorphism

$$\begin{aligned} \mathfrak{g}_r &\to \mathcal{T}(Z) \\ \epsilon &\mapsto \xi_\epsilon \end{aligned}$$

into the Lie algebra $\mathcal{T}(Z)$ of vector fields on Z. The homomorphism is defined by the relation

$$\xi_{\mathrm{Ad}\, g(\epsilon)} = dg \circ \xi_\epsilon \circ g^{-1}.$$

Given a basis $\{\epsilon_m\}$ for $\mathfrak{g}_r$, the vector fields ξ_{ϵ_m} are called the *generators* of a representation of the Lie group G on Z. Let $\mathfrak{g}^* = T^*_e G$ be the vector space dual of the tangent space T_1G, i.e. the dual Lie algbera. It is provided with the basis $\{\epsilon^m\}$ dual to the basis $\{\epsilon_m\}$ for T_1G. The group G and the right Lie algebra $\mathfrak{g}_r$ act on $\mathfrak{g}^*$ by the coadjoint representation

$$\mathrm{ad}^*_{\epsilon_m}(\epsilon^n) = -c^n_{mk}\epsilon^k.$$

A differential form ϕ on the Lie group G is said to be *left-invariant* if $L^*_g\phi = \phi$. The exterior derivative of a left-invariant form is again left-invariant. In particular, the left-invariant 1-forms satsify the Cartan equations

$$d\phi(\epsilon, \epsilon') = -\frac{1}{2}\phi([\epsilon, \epsilon']), \quad \epsilon, \epsilon' \in \mathfrak{g}_l.$$

We have a canonical $\mathfrak{g}_l$-valued left-invariant 1-form

$$\theta_l : T_1G \to \mathfrak{g}_l, \quad \theta_l(\epsilon) = \epsilon$$

on the Lie group G. With respect to the decomposition $\theta_l = \theta_l^m \epsilon_m$, the coefficients θ_l^m make up the basis for the space of left-invariant exterior 1-forms on G:

$$\epsilon_m \rfloor \theta_l^n = \delta_m^n.$$

The Cartan equations, written with respect to this basis, read

$$d\theta_l^m = \frac{1}{2} c_{nk}^m \theta_l^n \wedge \theta_l^k.$$

Analogous statements hold when one replaces "left" with "right".

The canonical action of G on P on the right defines the canonical trivial vertical splitting

$$\alpha : VP \xrightarrow{\cong} P \times \mathfrak{g}_l$$

such that $\alpha^{-1}(\epsilon_m)$ are the familiar fundamental vector fields on P corresponding to the basis elements ϵ_m of the Lie algebra $\mathfrak{g}_l$.

Taking the quotient of the tangent bundle $TP \to P$ and the vertical tangent bundle $VP \to P$ by dR_G (or by G) yields the vector bundles

$$T_GP = (TP)/G \to X \quad \text{and} \quad V_GP = VP/G \to X.$$

The fibers of T_GP encode both vertical directions (isomorphic to $\mathfrak{g}_l$) and the projection onto horizontal directions. The natural projection $T_GP \to \mathcal{T}(X)$ is given by differentiating the bundle projection $P \to X$.

Sections of $T_GP \to X$ are G-invariant vector fields on P, and sections of $V_GP \to X$ are G-invariant vertical vector fields on P. Hence, the typical fiber of $V_GP \to X$ is the right Lie algebra $\mathfrak{g}_r$ of the right-invariant vector fields on the group G. The group G acts on this typical fiber by the adjoint representation.

The Lie bracket of G-invariant vector fields on P descends to the quotient by G and defines the Lie bracket of sections of the vector bundles $T_GP \to X$ and $V_GP \to X$. It follows that $V_GP \to X$ is a Lie algebra bundle (called the *gauge algebra bundle*) whose fibers are Lie algebras isomorphic to the right Lie algebra $\mathfrak{g}_r$ of G.

Given a local trivialization of P, there are corresponding local bundle trivializations of T_GP and V_GP. Given the basis $\{\epsilon_p\}$ for $\mathfrak{g}_r$, we obtain the local fibers $\{\partial_\lambda, e_p\}$ for $T_GP \to X$ and $\{e_p\}$ for V_GP. If we look at sections

$$\xi = \xi^\lambda \partial_\lambda + \xi^p e_p, \quad \eta = \eta^\mu \partial_\mu + \eta^q e_q$$

of $T_GP \to X$, then we see that the coordinate expression of their bracket is

$$[\xi, \eta] = (\xi^\mu \partial_\mu \eta^\lambda - \eta^\mu \partial_\mu \xi^\lambda)\partial_\lambda + (\xi^\lambda \partial_\lambda \eta^r - \eta^\lambda \partial_\lambda \xi^r + c_{pq}^r \xi^p \eta^q) e_r. \qquad (1.2)$$

Given fiber bundles $\pi_1 : Y_1 \to X$ and $\pi_2 : Y_2 \to X$, their *fibered product* is defined as the pullback bundle

$$Y_1 \times_X Y_2 = \pi_1^* Y_2 = \pi_2^* Y_1 = \{(y_1, y_2) : \pi_1(y_1) = \pi_2(y_2)\}$$

which is of course another fiber bundle over X.

Let $\pi_Y : TY \to Y$ be the tangent bundle of a fiber bundle $\pi : Y \to X$. Given local coordinates (x^λ, y^i) on Y, the tangent bundle TY is equipped with the holonomic coordinates $(x^\lambda, y^i, \dot{x}^\lambda, \dot{y}^i)$. The tangent bundle $TY \to Y$ has the *vertical subbundle* $VY = \operatorname{Ker} d\pi$, consisting of the vectors tangent to the fibers of Y. The vertical tangent bundle VY is provided with the holonomic coordinates $(x^\lambda, y^i, \dot{y}^i)$ with respect to the frames $\{\partial_i\}$. The vertical cotangent bundle $V^*Y \to Y$ of the fiber bundle $Y \to X$ is defined as the vector bundle dual of the vertical tangent bundle $VY \to Y$.

A vector field u on a fiber bundle $Y \xrightarrow{\pi} X$ is said to be *projectable* if it projects over a vector field τ on X, i.e., if it satisfies $\tau \circ \pi = d\pi \circ u$. A projectable vector field has the coordinate expression

$$u = u^\lambda(x^\mu)\partial_\lambda + u^i(x^\mu, y^j)\partial_i, \quad \tau = u^\lambda \partial_\lambda.$$

A projectable vector field $u = u^i \partial_i$ on a fiber bundle $Y \to X$ is said to be *vertical* if it projects over the zero vector field $\tau = 0$ on X.

A vector field $\tau = \tau^\lambda \partial_\lambda$ on the base X can give rise to a projectable vector field on the total space Y by means of some connection on this fiber bundle. Nevertheless, any vector field τ on X admits a *canonical lift* to any tensor bundle, which we will denote by $\widetilde{\tau}$. In particular, the lift of τ onto the tangent bundle is given by

$$\widetilde{\tau} = \tau^\mu \partial_\mu + \partial_\nu \tau^\alpha \dot{x}^\nu \frac{\partial}{\partial \dot{x}^\alpha} \tag{1.3}$$

and the lift of τ onto the cotangent bundle is given by

$$\widetilde{\tau} = \tau^\mu \partial_\mu - \partial_\beta \tau^\nu \dot{x}_\nu \frac{\partial}{\partial \dot{x}_\beta}. \tag{1.4}$$

The Lie derivative of an exterior form ϕ along a vector field u is denoted by

$$\mathbf{L}_u \phi = u \rfloor d\phi + d(u \rfloor \phi).$$

Given the tangent lift $\widetilde{\phi}$ of an exterior form ϕ, we note that

$$\mathbf{L}_u \phi = u^* \widetilde{\phi}.$$

Given a differential form ω, we will use the notation

$$\omega_\lambda = \partial_\lambda \rfloor \omega, \qquad \omega_{\mu\lambda} = \partial_\mu \rfloor \partial_\lambda \rfloor \omega.$$

1.2. Jet manifolds. The order-k jet manifold J^kY of a fiber bundle $Y\to X$ consists of the equivalence classes j^k_xs, $x\in X$, of sections s of Y, identified by the first $k+1$ terms of their Taylor series at the points $x\in X$. The jet manifold J^kY is provided with the adapted coordinates

$$(x^\lambda, y^i, y^i_\lambda,\ldots, y^i_{\lambda_k,\cdots\lambda_1}), \quad y^i_{\lambda_l\cdots\lambda_1}(j^k_xs)=\partial_{\lambda_l}\cdots\partial_{\lambda_1}s^i(x),$$

for $0\leq l\leq k$.

We will use the following notation for operators on exterior forms on jet manifolds. The *total derivative* operator is defined by

$$d_\lambda=\partial_\lambda+y^i_\lambda\partial_i+y^i_{\lambda\mu}\partial^\mu_i+\cdots. \tag{1.5}$$

For instance, $d_\lambda(dx^\mu)=0$ and $d_\lambda(dy^i_{\lambda l\cdots\lambda_1})=dy^i_{\lambda\lambda_l\cdots\lambda_1}$. The horizontal projection h_0 is defined in local coordinates by

$$h_0(dy^i)=y^i_\mu dx^\mu, \qquad h_0(dy^i_\lambda)=y^i_{\mu\lambda}dx^\mu,$$

and similarly for the higher order jet manifolds. The *horizontal differential* is defined by

$$d_H\phi=dx^\lambda\wedge d_\lambda(\phi).$$

The total derivative satisfies the graded Leibniz rule

$$d_\lambda(\phi\wedge\sigma)=d_\lambda(\phi)\wedge\sigma+\phi\wedge d_\lambda(\sigma), \qquad d_\lambda\circ d=d\circ d_\lambda,$$

while the horizontal differential satisfies

$$d_H\circ d_H=0, \qquad h_0\circ d=d_H\circ h_0.$$

2. Principal connections

Let J^1P denote the first jet manifold of a G-principal bundle $P\to X$. Since $J^1P\to P$ is an affine bundle modeled over the vector bundle

$$T^*X\otimes VP\longrightarrow P,$$

we can pass to the quotient by the jet prolongation J^1R_G of the canonical right G-action on P. This yields the affine bundle

$$C=J^1P/G\longrightarrow X,$$

modeled over the associated vector bundle

$$\overline{C}=T^*X\otimes V_GP\longrightarrow X,$$

so that there is a canonical vertical splitting

$$VC \simeq C \times_X \overline{C}.$$

It follows that $J^1P \to C$ is itself a principal G-bundle, canonically isomorphic to the pullback

$$J^1P \simeq P_C = C \times_X P \longrightarrow C.$$

Now recall that the G-invariant exact sequence

$$0 \longrightarrow V_GP \hookrightarrow T_GP \longrightarrow TX \longrightarrow 0 \tag{2.1}$$

arises by taking the quotient of the standard short exact sequence $0 \to VP \to TP \to \pi^*TX \to 0$ with respect to the G-action. A *principal connection* on $P \to X$ is defined as a section

$$A : P \longrightarrow J^1P$$

which is G-equivariant in the sense that

$$J^1R_g \circ A = A \circ R_g \tag{2.2}$$

for any $g \in G$. Such a section determines a splitting of the exact sequence (2.1). In local coordinates, a principal connection is represented by a T_GP-valued 1-form

$$\begin{aligned} &A : X \longrightarrow T^*X \otimes T_GP, \\ &A = dx^\lambda \otimes \big(\partial_\lambda + A^q_\lambda e_q\big), \end{aligned} \tag{2.3}$$

where e_q is a local basis of V_GP and A^q_λ are local smooth functions on X.

Condition (2.2) ensures that principal connections on $P \to X$ are in one-to-one correspondence with global sections of the affine bundle $C \to X$. Since affine bundles always admit global sections, every principal bundle admits a principal connection.

For this reason, $C = J^1P/G \to X$ is called the *bundle of principal connections* on P. Given a local trivialization of P, the associated bundle coordinates (x^λ, a^q_λ) on C identify a section A with local functions $A^q_\lambda = a^q_\lambda \circ A$, which are precisely the coefficients of the connection form (2.3). In gauge theory, these coefficients are interpreted as *gauge potentials*, and we therefore refer to sections $A : X \to C$ as gauge potentials.

Suppose a principal connection on $P \to X$ is represented by the vertical-valued 1-form

$$A = (dy^i - A^i_\lambda \, dx^\lambda) \otimes \partial_i,$$

where y^i denote local fiber coordinates on P and ∂_i span the vertical tangent bundle VP. This form encodes the splitting of TP into horizontal and vertical subspaces: the terms $dy^i - A^i_\lambda dx^\lambda$ vanish on horizontal vectors, while the coefficients A^i_λ record how the horizontal lift depends on the base coordinates. Composing A with the natural isomorphism $\alpha : VP \simeq P \times \mathfrak{g}_l$, we obtain the familiar Lie algebra–valued connection

1-form
$$\overline{A} : P \xrightarrow{A} T^*P \otimes_P VP \xrightarrow{\mathrm{Id}\otimes\alpha} T^*P \otimes \mathfrak{g}_l.$$
Thus $\overline{A}$ is the $\mathfrak{g}_l$-valued 1-form on P that annihilates horizontal vectors and restricts to the Maurer–Cartan form on vertical directions.

With respect to a local trivialization (U_ζ, z_ζ) of P, the connection form has the local expression
$$\overline{A} = \psi_\zeta^*\big(\theta_l - \overline{A}^q_\lambda \, dx^\lambda \otimes \epsilon_q\big) \,, \tag{2.4}$$
where θ_l is the canonical left-invariant $\mathfrak{g}_l$-valued 1-form on G, $\{\epsilon_q\}$ is a basis of $\mathfrak{g}_l$, and the coefficients $\overline{A}^q_\lambda$ are smooth functions on P satisfying the equivariance condition
$$\overline{A}^q_\lambda(pg)(\epsilon_q) = \mathrm{ad}(g^{-1})\big(\overline{A}^q_\lambda(p)(\epsilon_q)\big) \,.$$
By defining local functions on X via $A^q_\lambda = \overline{A}^q_\lambda \circ z_\zeta$ over U_ζ, we recover the usual local connection 1-form
$$A_\zeta = -A^q_\lambda dx^\lambda \otimes \epsilon_q = A^q_\lambda dx^\lambda \otimes \epsilon_q, \tag{2.5}$$
which is just the pull-back $z_\zeta^*\overline{A}$ of the connection 1-form $\overline{A}$ over U_ζ. Note that these coefficients agree with those appearing in the expression (2.3) for the splitting of T_GP. It is important to note that, by virture of the natural identification $V_GP \simeq P \times_G \mathfrak{g}_l$, the coefficients A^q_λ admit two interpretations:

- In (2.3) they occur in the V_GP-valued 1-form
$$A^q_\lambda \, dx^\lambda \otimes e_q \in \Omega^1(X) \otimes V_GP,$$
where $\{e_q\}$ denotes the basis of V_GP induced from $\{\epsilon_q\}$;
- In (2.5) they appear in the $\mathfrak{g}_l$-valued local connection 1-form.

Principal connections behave well under morphisms of bundles. We record two basic functorial properties.

Fact 2.1 (Pullback)**.** *Let $P \to X$ be a principal G-bundle and $f : Y \to X$ a smooth map. Then the pullback $f^*P \to Y$ is again a principal G-bundle, and the canonical morphism $f_P : f^*P \to P$ allows one to pull back a connection A on P to a connection f^*A on f^*P. Concretely, the horizontal distribution of f^*A is defined as the preimage under df_P of the horizontal distribution of A.*

Fact 2.2 (Pushforward)**.** *Let $P' \to X$ and $P \to X$ be principal bundles with structure groups G' and G, respectively, and let $\Phi : P' \to P$ be a principal bundle morphism covering Id_X and corresponding to a homomorphism $G' \to G$. If A' is a principal connection on P', then there exists a unique principal connection A on P such that $d\Phi$ maps horizontal subspaces of A' onto those of A. In this way, Φ pushes forward connections from P' to P.*

2.1. Curvature. The curvature of a principal connection A (also called the *field strength*) is defined as the V_GP-valued 2-form on X,

$$\begin{aligned} F_A &: X \longrightarrow \Lambda^2 T^*X \otimes V_GP, \\ F_A &= \tfrac{1}{2}\, F^r_{\lambda\mu}\, dx^\lambda \wedge dx^\mu \otimes e_r. \end{aligned} \tag{2.6}$$

In local coordinates, the coefficients are computed from the commutator of horizontal lifts:

$$F^r_{\lambda\mu} = \left[\partial_\lambda + A^p_\lambda e_p, \partial_\mu + A^q_\mu e_q\right]^r = \partial_\lambda A^r_\mu - \partial_\mu A^r_\lambda + c^r_{pq}\, A^p_\lambda A^q_\mu, \tag{2.7}$$

where c^r_{pq} are the structure constants of $\mathfrak{g}_l$. Equivalently, if we regard the local connection form as the V_GP-valued 1-form

$$A = A^q_\lambda\, dx^\lambda \otimes e_q,$$

then the curvature takes the familiar shape

$$F_A = dA + A \wedge A,$$

with the wedge product understood using the Lie bracket on V_GP. It is important to distinguish F_A from two related notions:

1. Curvature of affine connections: for a connection on a vector bundle, the curvature is a 2-form with values in $\mathrm{End}(E)$. Here, F_A is not an endomorphism-valued form but rather a section of $\Lambda^2 T^*X \otimes V_GP$, reflecting the fact that a principal connection splits the sequence (2.1) and its curvature measures the non-integrability of the corresponding horizontal distribution.

2. Lie algebra–valued curvature form: composing with the natural isomorphism $\alpha : V_GP \xrightarrow{\simeq} P \times_G \mathfrak{g}_l$ gives the $\mathfrak{g}_l$-valued curvature 2-form on P

$$\Omega = d\overline{A} \;+\; \tfrac{1}{2}[\overline{A}, \overline{A}] \in \Lambda^2 T^*P \otimes \mathfrak{g}_l,$$

   where $\overline{A}$ is the connection 1-form (2.4) on P. These two curvatures are related by the rule

$$z^*_\zeta \Omega = -\,\psi_\zeta(F_A)$$

   on any local trivialization (U_ζ, ψ_ζ).

In practice, one often passes to the local $\mathfrak{g}_l$-valued 2-form

$$\psi_\zeta(F_A) = dA_\zeta + \tfrac{1}{2}[A_\zeta, A_\zeta],$$

where $A_\zeta = A^q_\lambda\, dx^\lambda \otimes \epsilon_q$ is the local connection 1-form (2.5). This form has the same component expression as (2.6), but written with respect to the Lie algebra basis $\{\epsilon_r\}$ instead of the induced vertical basis $\{e_r\}$. Thus the curvature can be viewed either as a V_GP-valued 2-form on X or, after trivialization, as the familiar $\mathfrak{g}_l$-valued field strength.

2.2. Associated bundles. Let $Y = (P \times V)/G \to X$ be the fiber bundle associated with the principal G-bundle $P \to X$, where G acts on the typical fiber V on the left. We call such a bundle a *P-associated bundle*. By definition, the quotient is formed by identifying

$$(p, v) \sim (pg,\, g^{-1}v), \qquad p \in P, g \in G, v \in V.$$

For each $p \in P$, the map

$$[p] : V \to Y_{\pi(p)}, \qquad [p](v) = (p, v) \cdot G,$$

is a linear isomorphism onto the fiber over $\pi(p) \in X$, and satisfies the compatibility condition

$$[p](v) = [pg](g^{-1}v).$$

Every principal connection on $P \to X$ canonically induces a connection on any P-associated bundle Y. Indeed, given a connection A on P with horizontal distribution $H \subset TP$, the differential of the projection

$$P \times V \longrightarrow (P \times V)/G = Y$$

sends $H \times V$ to a horizontal distribution in TY, thereby defining a connection on $Y \to X$. We will call this induced connection the *associated connection*.

If Y is in fact a vector bundle associated to a representation $\rho : G \to \mathrm{Aut}(V)$, then the induced connection admits the local form

$$A = dx^\lambda \otimes \Big(\partial_\lambda - A^p_\lambda\, I_p{}^i\, \partial_i\Big),$$

where $\{I_p\}$ are the infinitesimal generators of the representation $d\rho : \mathfrak{g}_l \to \mathrm{End}(V)$. Explicitly, if $\{\epsilon_p\}$ is a basis of $\mathfrak{g}_l$, then $I_p = d\rho(\epsilon_p)$ describes how ϵ_p acts on V. These matrices encode the geometric action of the structure group on the fiber. In this case, the curvature of this associated connection reads

$$F = -\tfrac{1}{2} F^p_{\lambda\mu}\, I_p{}^i\, dx^\lambda \wedge dx^\mu \otimes \partial_i,$$

with the same local coefficients $F^p_{\lambda\mu}$ as in (2.6).

In particular, applying this construction to the adjoint representation of G yields the *gauge algebra bundle*

$$V_G P \simeq P \times_G \mathfrak{g}_l \longrightarrow X.$$

The induced connection on $V_G P$ is a linear connection, whose covariant differential of a section $\xi = \xi^p e_p$ is

$$\begin{aligned} &\nabla^A \xi : X \to T^*X \otimes V_G P, \\ &\nabla^A \xi = \big(\partial_\lambda \xi^r + c^r_{pq} A^p_\lambda\, \xi^q\big)\, dx^\lambda \otimes e_r. \end{aligned} \tag{2.8}$$

For a vector field u on X, the covariant derivative takes the form

$$\nabla^A_u \xi = u \rfloor \nabla^A \xi = [\, u \rfloor A, \xi \,],$$

where A is the $T_G P$-valued 1-form (2.3). In particular,

$$\nabla^A_{\partial_\lambda} e_q = c^r_{pq} A^p_\lambda e_r.$$

Finally, this covariant derivative is compatible with the Lie bracket on sections of $V_G P$, in the sense that

$$\nabla^A_u [\, \xi, \eta \,] = [\, \nabla^A_u \xi,\, \eta \,] + [\, \xi,\, \nabla^A_u \eta \,],$$

for any vector field u and sections ξ, η of $V_G P \to X$.

2.3. The Nijenhuis differential. Given a smooth manifold M, let $\mathfrak{O}^r(M)$ denote the space of differential r-forms on M. The direct sum

$$\mathfrak{O}(M) = \bigoplus_{r \geq 0} \mathfrak{O}^r(M)$$

is the $\mathbb{Z}$-graded exterior algebra with respect to the wedge product $\wedge$. The exterior differential acts by

$$\begin{aligned} &d : \mathfrak{O}^r(M) \to \mathfrak{O}^{r+1}(M), \\ &d\phi = \frac{1}{r!}\, \partial_\mu \phi_{\lambda_1 \cdots \lambda_r}\, dz^\mu \wedge dz^{\lambda_1} \wedge \cdots \wedge dz^{\lambda_r}. \end{aligned}$$

The space of tangent-valued forms $\mathfrak{O}^*(M) \otimes \mathcal{T}(M)$ carries a natural graded Lie bracket, the *Frölicher–Nijenhuis bracket (FN bracket)*, which extends the Lie bracket of vector fields:

$$[\cdot,\cdot]_{FN} : \mathfrak{O}^r(M) \otimes \mathcal{T}(M) \times \mathfrak{O}^s(M) \otimes \mathcal{T}(M) \longrightarrow \mathfrak{O}^{r+s}(M) \otimes \mathcal{T}(M) \tag{2.9}$$

which is determined on decomposable elements by

$$\begin{aligned} [\alpha \otimes u, \beta \otimes v]_{FN} = {} & (\alpha \wedge \beta) \otimes [u, v] + (\alpha \wedge \mathbf{L}_u \beta) \otimes v - (\mathbf{L}_v \alpha \wedge \beta) \otimes u \\ & + (-1)^r (d\alpha \wedge u \rfloor \beta) \otimes v + (-1)^r (v \rfloor \alpha \wedge d\beta) \otimes u, \end{aligned}$$

for $\alpha \in \mathfrak{O}^r(M)$, $\beta \in \mathfrak{O}^s(M)$ and $u, v \in \mathcal{T}(M)$. For $r = s = 0$ this reduces to the usual Lie bracket of vector fields.

Now let $P \to X$ be a principal G-bundle. The FN bracket on $\mathfrak{O}^*(P) \otimes \mathcal{T}(P)$ is G-equivariant with respect to the canonical right action R_G, meaning that $[\cdot,\cdot]_{FN}$ commutes with the pullback action of R_g for all $g \in G$. Consequently, it descends to an induced FN bracket on the associated bundle $T_G P = (TP)/G \to X$.

Let $A \in \mathfrak{O}^1(X) \otimes T_G P$ be the local connection form of a principal connection, as in (2.5). The associated *Nijenhuis differential* is defined by

$$\begin{aligned} d_A &: \mathfrak{O}^r(X) \otimes T_G P \longrightarrow \mathfrak{O}^{r+1}(X) \otimes V_G P, \\ d_A\phi &= [A, \phi]_{FN}, \end{aligned} \tag{2.10}$$

for $\phi \in \mathfrak{O}^r(X) \otimes T_G P$. On the vertical subbundle $V_G P$, this reduces to the covariant differential ∇^A introduced earlier in (2.8), i.e.

$$d_A\xi = \nabla^A\xi, \quad \xi \in \Gamma(V_G P).$$

Equivalently, in local form,

$$\nabla^A\xi = d\xi + [A, \xi],$$

where A is the vertical connection 1-form (2.3).

For decomposable elements $\phi = \alpha \otimes \xi$ with $\alpha \in \mathfrak{O}^r(X)$ and $\xi \in V_G P$, one obtains the graded Leibniz rule

$$d_A(\alpha \otimes \xi) = d\alpha \otimes \xi + (-1)^r\, \alpha \wedge \nabla^A\xi,$$

and the extension to general tensors follows by linearity.

Finally, the curvature of A can be expressed in terms of d_A and the FN bracket as

$$F_A = \tfrac{1}{2}\, d_A A + \tfrac{1}{2}[A, A]_{FN} \in \mathfrak{O}^2(X) \otimes V_G P. \tag{2.11}$$

This coincides with the usual definition $F_A = dA + \frac{1}{2}[A, A]$ when restricted to the vertical part.

2.4. The bundle of principal connections. We now turn to vector fields and connections on the bundle of principal connections $C = J^1P/G \to X$ which parametrizes all principal connections on $P \to X$. In particular, we will define a canonical connection on the pullback G-principal bundle $P_C = C \times_X P \to C$.

Let J^1P be the first jet bundle of P, with adapted coordinates $(x^\lambda, y^i, y^i_\lambda)$. There is a canonical bundle morphism

$$\theta : J^1P \times_P TP \longrightarrow VP,$$

which assigns to a tangent vector its vertical component relative to a chosen 1-jet of P. Passing to the quotient by G, this descends to a bundle morphism over X

$$\begin{aligned} &\theta : C \times_X T_G P \longrightarrow V_G P, \\ &\theta(\partial_\lambda) = -a^p_\lambda e_p, \qquad \theta(e_p) = e_p, \end{aligned} \tag{2.12}$$

where the coordinates a^p_λ describe a connection in C. Moreover, since

$$V_G(C \times_X P) = C \times_X V_G P, \qquad T_G(C \times_X P) = TC \times_X T_G P,$$

the Atiyah exact sequence for the pullback bundle $P_C \to C$ takes the form

$$0 \longrightarrow C \times_X V_G P \hookrightarrow TC \times_X T_G P \longrightarrow TC \longrightarrow 0. \tag{2.13}$$

The morphism (2.12) provides a canonical splitting of this exact sequence: it projects tangent vectors in $TC \times_X T_G P$ onto their vertical component in $C \times_X V_G P$, and thereby determines a complementary horizontal subspace.

Explicitly, this yields a horizontal splitting

$$TC \times_X T_G P \longrightarrow C \times_X T_G P \longrightarrow C \times_X V_G P,$$

and hence a canonical (but generally non-flat) principal connection

$$\mathcal{A} \in \mathfrak{O}^1(C) \otimes T_G(C \times_X P)$$

on $P_C \to C$, given in local form by

$$\begin{aligned} &\mathcal{A} : TC \to TC \times_X T_G P, \\ &\mathcal{A} = dx^\lambda \otimes (\partial_\lambda + a^p_\lambda e_p) + da^r_\lambda \otimes \partial^\lambda_r. \end{aligned}$$

Thus P_C is canonically equipped with the connection $\mathcal{A}$.

Consequently, the vector bundle $C \times_X V_G P \to C$ inherits a canonical linear connection with associated covariant differential given by

$$\partial_\lambda \lrcorner \nabla^{\mathcal{A}} e_q = c^r_{pq} a^p_\lambda e_r, \qquad \partial^\lambda_r \lrcorner \nabla^{\mathcal{A}} e_q = 0. \tag{2.14}$$

By definition (2.11), the curvature 2-form $F_{\mathcal{A}} \in \mathfrak{O}^2(C) \otimes V_G P$ is

$$\begin{aligned} F_{\mathcal{A}} &= \tfrac{1}{2}\, d_{\mathcal{A}} \mathcal{A} + \tfrac{1}{2}\, [\mathcal{A}, \mathcal{A}]_{FN} \\ &= \Big(da^r_\mu \wedge dx^\mu + \tfrac{1}{2} c^r_{pq} a^p_\lambda a^q_\mu dx^\lambda \wedge dx^\mu \Big) \otimes e_r. \end{aligned} \tag{2.15}$$

We note the following: if $A : X \to C$ is a principal connection on $P \to X$, then the curvature of A is exactly the pullback $F_A = A^* F_{\mathcal{A}}$.

Example 2.3 (Trivial principal bundle)**.** Let $P = X \times \mathbb{R} \to X$ be the trivial principal bundle with abelian structure group $(\mathbb{R}, +)$. Then $C = T^*X \to X$ is the affine cotangent bundle, and principal connections on P correspond exactly to 1-forms on X.

Choose local coordinates (x^λ) on X and (y^i) on the fiber. Then J^1P has local coordinates $(x^\lambda, y^i, \dot{x}^\lambda)$, and the right action by $t \in \mathbb{R}$ is

$$(x, y, \dot{x}^\lambda) \cdot t = (x, y + t, \dot{x}^\lambda).$$

Hence the G-invariants are $(x^\lambda, \dot{x}^\lambda)$, so

$$C = J^1P/G \cong \{(x^\lambda, a_\lambda)\}, \qquad a_\lambda = \dot{x}^\lambda.$$

Identifying C with T^*X via

$$\Psi : C \to T^*X$$
$$\Psi(x, a) = (x, \dot{x}_\lambda = a_\lambda),$$

then the tautological 1-form on T^*X, $\dot{x}_\lambda\, dx^\lambda$, pulls back to

$$\Psi^*(\dot{x}_\lambda\, dx^\lambda) = a_\lambda\, dx^\lambda.$$

On $J^1P \times_P TP$ the canonical morphism θ takes the vertical part of a tangent vector relative to a chosen 1-jet. In coordinates we have

$$\theta(\partial_\lambda) = -\dot{x}^\lambda\, \partial_y, \qquad \theta(\partial_y) = \partial_y.$$

Taking the quotient by G and using $a_\lambda = \dot{x}^\lambda$ yields

$$\theta(\partial_\lambda) = -a_\lambda\, e, \qquad \theta(e) = e,$$

where e is the class of ∂_y in $V_GP \simeq X \times \mathbb{R}$, in agreement with (2.12).

Take the local coordinates $(x^\lambda, a_\mu; y)$ on $P_C = C \times_X P$. Using θ to project vertical parts defines the horizontal lifts

$$(\partial_\lambda)^H = \partial_\lambda + a_\lambda\, e, \qquad (\partial^\mu)^H = \partial^\mu = \frac{\partial}{\partial a_\mu}.$$

Hence the canonical principal connection $\mathcal{A} \in \mathfrak{O}^1(C) \otimes T_G(C \times_X P)$ is

$$\mathcal{A} = dx^\lambda \otimes (\partial_\lambda + a_\lambda e) + da_\mu \otimes \partial^\mu$$

with vertical component (projection to V_GP) given by the universal potential

$$A_{\text{can}} = a_\lambda\, dx^\lambda \otimes e = \Psi^*\theta \otimes e.$$

Now we will compute the Nijenhuis differential $d_{\mathcal{A}}$ and the curvature $F_{\mathcal{A}}$. Since G is abelian, the structure constants vanish and $\mathrm{ad} = 0$. From (2.8) (with $c^r_{pq} = 0$) we get

$\nabla^{\mathcal{A}} e = 0$, and therefore for any scalar form f,

$$d_{\mathcal{A}}(f\,e) = df \otimes e.$$

Moreover $[A_{\mathrm{can}}, A_{\mathrm{can}}]_{FN} = 0$. Thus the curvature (2.15) becomes

$$\begin{aligned} F_{\mathcal{A}} &= d_{\mathcal{A}} A_{\mathrm{can}} = d(a_\lambda\, dx^\lambda) \otimes e = (da_\lambda \wedge dx^\lambda) \otimes e \\ &= d\dot{x}_\lambda \wedge dx^\lambda \otimes e = d\theta \otimes e. \end{aligned}$$

Under the identification $C \simeq T^*X$ this is precisely the canonical symplectic form $d\theta$.

We can relate the connection to the Lie-algebra valued form. Composing with $\alpha : V_G P \to \mathfrak{g}_l \simeq \mathbb{R}$, $\alpha(e) = \epsilon$, the $\mathfrak{g}_l$-valued connection on P_C is

$$\overline{\mathcal{A}} = dy - a_\lambda\, dx^\lambda, \qquad \Omega_{\mathcal{A}} = d\overline{\mathcal{A}} = -\, da_\lambda \wedge dx^\lambda.$$

Pulling back by the identity section $s : C \to P_C$, $s(x, a) = (x, a; 0)$, gives

$$s^*\overline{\mathcal{A}} = -\, a_\lambda\, dx^\lambda = -\, \Psi^*\theta, \qquad s^*\Omega_{\mathcal{A}} = -\, d\theta,$$

so that $\alpha(F_{\mathcal{A}}) = -\, \Omega_{\mathcal{A}}$, consistent with the general sign relation discussed earlier.

To summarize, we have:

$$\begin{aligned} &\theta = \dot{x}_\lambda\, dx^\lambda, \\ &d_{\mathcal{A}} = d, \\ &F_{\mathcal{A}} = d\dot{x}_\lambda \wedge dx^\lambda = d\theta \in \Lambda^2 T^*X. \end{aligned}$$

Thus $\mathcal{A}$ reproduces the tautological 1-form θ on T^*X, and its curvature $F_{\mathcal{A}}$ is the canonical symplectic form $d\theta$. This example illustrates that the bundle $C \to X$ in a sense generalizes the cotangent bundle: just as T^*X carries a canonical symplectic 2-form $d\theta$, the bundle of principal connections carries the canonical $V_G P$-valued 2-form (2.15). In particular, for a vector field u on X, the canonical lift $\widetilde{u}$ to T^*X is determined by

$$\widetilde{u} \rfloor d\theta = d(u \rfloor \theta).$$

In gauge theory this equation is generalized by means of the canonical curvature $F_{\mathcal{A}}$. ◇

Let $\xi = \tau^\lambda \partial_\lambda + \xi^p e_p$ be a section of the fiber bundle $T_G P \to X$ which projects onto a vector field τ on X. One can think of ξ as being a generator of a 1-parameter group of general gauge transformations of the principal bundle $P \to X$. Using (2.12) we obtain a morphism over X

$$\xi \rfloor \theta : C \to V_G P$$

which may be regarded as a section of $V_G(C \times_X P) \to C$. Then the equation

$$\xi_C \rfloor F_{\mathcal{A}} = d_{\mathcal{A}}(\xi \rfloor \theta)$$

uniquely determines a vector field ξ_C on C which also projects to τ. Let us expand this relation in coordinates. Write

$$\xi_C = \tau^\lambda \partial_\lambda + u^r_\lambda \partial^\lambda_r$$

for some coefficients u^r_λ to be determined. Contracting $F_\mathcal{A}$ from (2.15) with ξ_C gives

$$\xi_C \lrcorner F_\mathcal{A} = \left(\tau^\mu da^r_\mu + u^r_\mu dx^\mu + c^r_{pq} a^p_\mu \tau^\mu dx^\lambda a^q_\lambda\right) \otimes e_r.$$

On the other hand,

$$d_\mathcal{A}(\xi \lrcorner \theta) = \left(\partial_\lambda \xi^r + c^r_{pq} a^p_\lambda \xi^q - a^r_\mu \partial_\lambda \tau^\mu\right) dx^\lambda \otimes e_r.$$

Equating coefficients yields

$$u^r_\lambda = \partial_\lambda \xi^r + c^r_{pq} a^p_\lambda \xi^q - a^r_\mu \partial_\lambda \tau^\mu,$$

and therefore

$$\begin{aligned} &\xi_C = \tau^\lambda \partial_\lambda + u^r_\lambda \partial^\lambda_r, \text{ with} \\ &u^r_\lambda = \partial_\lambda \xi^r + c^r_{pq} a^p_\lambda \xi^q - a^r_\mu \partial_\lambda \tau^\mu. \end{aligned} \tag{2.16}$$

The vector field ξ_C is the generator of the associated gauge transformations of the bundle of principal connections C. In the special case $\xi \in V_G P$ (i.e. $\tau = 0$), this reduces to the vertical vector field

$$\xi_C = u^r_\lambda \partial^\lambda_r, \qquad u^r_\lambda = \partial_\lambda \xi^r + c^r_{pq} a^p_\lambda \xi^q.$$

Since $VC = C \times_X T^*X \otimes V_G P \subset TC$, we can equivalently write

$$\xi_C = \nabla^\mathcal{A} \xi : C \to VC.$$

Example 2.4. Let A be a principal connection on $P \to X$. For any vector field τ on X, consider the section

$$\begin{aligned} &\xi = \tau \lrcorner A : X \to T_G P, \\ &\xi = \tau^\lambda \partial_\lambda + A^p_\lambda \tau^\lambda e_p. \end{aligned}$$

The construction above yields the induced vector field (2.16) on C,

$$\begin{aligned} &\widetilde{\tau}_A = \tau^\lambda \partial_\lambda + u^r_\lambda \partial^\lambda_r, \\ &u^r_\lambda = \partial_\lambda A^r_\mu \tau^\mu + c^r_{pq} a^p_\lambda A^q_\mu \tau^\mu - (a^r_\mu - A^r_\mu) \partial_\lambda \tau^\mu. \end{aligned}$$

This example shows that, once a background principal connection A is chosen, vector fields on X induce induce vector fields on the bundle of connections C in a natural way. ◇

Returning to the curvature $F_\mathcal{A}$, we note that it can be viewed in a slightly different way. Namely, there is a canonical horizontal V_GP-valued 2-form on J^1C,

$$\mathcal{F} = \tfrac{1}{2}\,\mathcal{F}^r_{\lambda\mu}\,dx^\lambda \wedge dx^\mu \otimes e_r,$$
$$\mathcal{F}^r_{\lambda\mu} = a^r_{\lambda\mu} - a^r_{\mu\lambda} + c^r_{pq}\,a^p_\lambda a^q_\mu,$$

where $(x^\lambda, a^p_\mu, a^p_{\lambda\mu})$ are adapted coordinates on J^1C. Now, for any principal connection $A : X \to C$, we let $J^1A : X \to J^1C$ denote the first jet prolongation, i.e., $J^1A(x) = j^1_xA$ is the 1-jet of A at x. Evaluating $\mathcal{F}$ on this jet recovers the horizontal part of the curvature of A:

$$\mathcal{F} \circ J^1A = h_0(F_A),$$

i.e. the components $\mathcal{F}^r_{\lambda\mu}(j^1_xA)$ are exactly the components of F_A expressed as a horizontal 2-form on X. Moreover, note that the map

$$\frac{\mathcal{F}}{2} : J^1C \longrightarrow C \times_X \Lambda^2T^*X \otimes V_GP$$

is an affine surjection over C. To see this: $J^1C \to C$ is an affine bundle modeled on the vector bundle $T^*X \otimes_C VC$, and in coordinates the dependence of $\mathcal{F}^r_{\lambda\mu}$ on the fiber coordinates $a^r_{\lambda\mu}$ is affine (indeed linear) and the remaining terms $c^r_{pq}a^p_\lambda a^q_\mu$ depend only on the base coordinates (x^λ, a^s_ν) on C. Hence, for fixed $(x, a) \in C$ the map

$$(a^r_{\lambda\mu}) \longmapsto \mathcal{F}^r_{\lambda\mu}$$

is an affine map from the fiber $J^1C|_x$ onto the space $\Lambda^2T^*_xX \otimes V_GP_x$. In fact, this map is an affine surjection, because the antisymmetric part of the $a^r_{\lambda\mu}$-variables may be chosen arbitrarily to realize any target 2-form. Thus $\mathcal{F}/2$ is an affine surjection over C.

Therefore the kernel $C_+ = \operatorname{Ker}\mathcal{F}$ is an affine subbundle of $J^1C \to C$, and we obtain a canonical splitting over C:

$$J^1C = C_+ \oplus C_- = C_+ \oplus (C \times_X \Lambda^2T^*X \otimes V_GP). \tag{2.17}$$

The corresponding projections are $\mathrm{pr}_2 = \mathcal{F}/2$, and $\mathrm{pr}_1 = \mathcal{S}$ given by

$$\mathrm{pr}_1 = \mathcal{S} : J^1C \to C_+$$
$$S^r_{\lambda\mu} = \frac{1}{2}(a^r_{\lambda\mu} + a^r_{\mu\lambda} - c^r_{pq}a^p_\lambda a^q_\mu),$$

which extracts the symmetric (in λ, μ) part corrected by the quadratic term. Finally, if $\Gamma : C \to J^1C$ is a connection on the bundle of principal connections $C \to X$, then $\mathcal{S} \circ \Gamma$ is a C_+-valued connection on $C \to X$, meaning it satisfies the condition $\mathcal{F}(\mathcal{S} \circ \Gamma) = 0$. Writing this condition out in local coordinates yields

$$\mathcal{F}^r_{\lambda\mu}\big((\mathcal{S} \circ \Gamma)(x)\big) = (\mathcal{S} \circ \Gamma)^r_{\lambda\mu} - (\mathcal{S} \circ \Gamma)^r_{\mu\lambda} + c^r_{pq}a^p_\lambda a^q_\mu = 0. \tag{2.18}$$

In words: the antisymmetric part of the second-order jet coordinates of $\mathcal{S}\circ\Gamma$ is determined by the quadratic term $-c^r_{pq}a^p_\lambda a^q_\mu$, ensuring that $\mathcal{S}\circ\Gamma$ takes values in the kernel of $\mathcal{F}$.

3. Gauge conservation laws

Conservation laws in the gauge theory of principal connections exhibit the following fundamental features.

- In general, Noether conservation laws and Noether currents depend on gauge parameters, but this is not the case in an abelian gauge model (principal bundle with abelian structure group).
- Noether currents reduce to superpotentials because generators of gauge transformations depend on derivatives of gauge parameters.
- An energy-momentum conservation law implies the gauge invariance of a Lagrangian.

3.1. Lagrangian field theory. We will discuss Lagrangians, Euler–Lagrange operators, and conserved currents, following [T94, NS96, CVB03, H17]. The Lagrangian plays an important role for several reasons:

- The Euler–Lagrange operator $\mathcal{E}_L$ associated with a Lagrangian L governs the field equations (its kernel defines solutions).
- Symmetry currents are obtained from invariance properties of L.
- Conservation laws arise when symmetry currents interact with the Euler–Lagrange equations.

We follow the geometric formulation [CVB03] of classical field theory, where fields are represented by sections of a configuration bundle. For example, matter fields, gauge fields, gravitational fields all fit into this framework. We do not specify the type of fields, instead using y^i to denote all of them. The finite-dimensional configuration space of fields is the first order jet manifold J^1Y of $Y\to X$, with coordinates $(x^\lambda, y^i, y^i_\lambda)$, cf. [CVB03].

A *first-order Lagrangian* is defined as a horizontal density on J^1Y,

$$\begin{aligned} &L : J^1Y \to \Lambda^n T^*X \\ &L = \mathcal{L}(x^\lambda, y^i, y^i_\lambda)\omega \qquad \omega = dx^1\wedge\cdots\wedge dx^n. \end{aligned}$$

where $n=\dim X$. The function $\mathcal{L}(x^\lambda, y^i, y^i_\lambda)$ is any smooth real-valued function on the first jet manifold J^1Y. By definition $\mathcal{L}$ depends only on $(x^\lambda, y^i, y^i_\lambda)$, and not any higher jets. The associated Lagrangian is the horizontal n-form

$$L=\mathcal{L}\,\omega, \qquad \omega = dx^1\wedge\cdots\wedge dx^n.$$

In applications one often imposes further conditions on $\mathcal{L}$, such as:

- *Invariance:* $\mathcal{L}$ may be required to be covariant under diffeomorphisms of X or gauge transformations of Y.

- *Regularity:* non-degeneracy of the Hessian

$$\frac{\partial^2 \mathcal{L}}{\partial y^i_\lambda \partial y^j_\mu}$$

  is often important, especially in the context of Hamiltonian gauge theory.

Let $u = u^\lambda(x,y)\partial_\lambda + u^i(x,y)\partial_i$ be a projectable vector field on $Y \to X$. Its prolongation to the jet bundle is

$$J^1 u = u^\lambda \partial_\lambda + u^i \partial_i + (d_\lambda u^i - y^i_\mu \partial_\lambda u^\mu)\partial^\lambda_i.$$

The Lie derivative of L along $J^1 u$ is then

$$\mathbf{L}_{J^1 u} L = \Big[\partial_\lambda u^\lambda\, \mathcal{L} + (u^\lambda \partial_\lambda + u^i \partial_i + (d_\lambda u^i - y^i_\mu \partial_\lambda u^\mu)\partial^\lambda_i)\mathcal{L}\Big]\omega. \tag{3.1}$$

The *first variational formula* provides a canonical decomposition of this Lie derivative, separating the Euler–Lagrange term from a horizontally exact term:

$$\begin{aligned}\mathbf{L}_{J^1 u} L &= u_V \lrcorner\, \mathcal{E}_L + d_H h_0(u \lrcorner\, H_L)\\ &= (u^i - y^i_\mu u^\mu)(\partial_i - d_\lambda \partial^\lambda_i)\mathcal{L}\,\omega - d_\lambda\big[\pi^\lambda_i(u^\mu y^i_\mu - u^i) - u^\lambda \mathcal{L}\big]\omega \end{aligned} \tag{3.2}$$

where $u_V = (u \lrcorner\, \theta^i)\partial_i$ is the vertical part of u relative to the canonical splitting

$$u^\lambda \partial_\lambda + u^i \partial_i = u^\lambda(\partial_\lambda + y^i_\lambda \partial_i) + (u^i - u^\lambda y^i_\lambda)\partial_i.$$

Recall that we defined the operators d_λ, h_0, and d_H around (1.5).

The *Euler–Lagrange operator* associated to L is

$$\begin{aligned}&\mathcal{E}_L : J^2 Y \to T^*Y \wedge \Lambda^n T^*X,\\ &\mathcal{E}_L = (\partial_i \mathcal{L} - d_\lambda \pi^\lambda_i)\,\theta^i \wedge \omega,\end{aligned} \tag{3.3}$$

where we set $\pi^\lambda_i = \partial^\lambda_i \mathcal{L}$ for brevity. The associated *Poincaré–Cartan form* is

$$\begin{aligned}&H_L : J^1 Y \to T^*Y \wedge \Lambda^{n-1} T^*X,\\ &H_L = L + \pi^\lambda_i \theta^i \wedge \omega_\lambda = \pi^\lambda_i dy^i \wedge \omega_\lambda + (\mathcal{L} - \pi^\lambda_i y^i_\lambda)\omega.\end{aligned} \tag{3.4}$$

Here $\omega_\lambda = \partial_\lambda \lrcorner\, \omega$ is the contraction of ω with ∂_λ.

Remark 3.1 (Notation)**.** The shorthand $\pi^\lambda_i = \partial^\lambda_i \mathcal{L}$ denotes the partial derivative of the Lagrangian function $\mathcal{L}$ with respect to the jet coordinate y^i_λ. Thus, π^λ_i are the *generalized momenta* conjugate to the field components y^i, and they play a central role both in the Euler–Lagrange operator (3.3) and in the definition of the Poincaré–Cartan form (3.4). ◇

3.2. Conservation laws. The kernel of the Euler-Lagrange operator $\mathcal{E}_L$ is defined in local coordinates by the relations

$$(\partial_i - d_\lambda \partial_i^\lambda)\mathcal{L} = 0. \tag{3.5}$$

These equations define the system of second-order *Euler-Lagrange equations.* Classical solutions of these equations are sections s of the fiber bundle $X \to Y$ whose second order jet prolongations $J^2 s$ satisfy (3.5). Expanded in more detail, these equations are

$$\partial_i \mathcal{L} \circ s - (\partial_\lambda + \partial_\lambda s^j \partial_j + \partial_\lambda \partial_\mu s^j \partial_j^\mu)\partial_i^\lambda \mathcal{L} \circ s = 0. \tag{3.6}$$

The first variational formula (3.2) provides the standard procedure for studying the differential conservation laws in Lagrangian field theory. Let u be a projectable vector field on a fiber bundle $Y \to X$, treated as the generator of a local 1-parameter group of gauge transformations. Introduce the Euler–Lagrange components

$$\delta_i \mathcal{L} = \partial_i \mathcal{L} - d_\lambda \pi_i^\lambda = (\partial_i - d_\lambda \partial_i^\lambda)\mathcal{L}.$$

Substituting this into (3.6) gives the decomposition

$$\mathbf{L}_{J^1 u} L = (u^i - y_\mu^i u^\mu)\, \delta_i(\mathcal{L})\, \omega - d_\lambda \mathcal{I}^\lambda\, \omega, \tag{3.7}$$

where

$$\mathcal{I}^\lambda = \pi_i^\lambda (u^\mu y_\mu^i - u^i) - u^\lambda \mathcal{L}$$

are the components of the *symmetry current* $\mathcal{I} = \mathcal{I}^\lambda \omega_\lambda$ associated to u. We now introduce the weak equality $\Phi \approx \Psi$ to denote equality modulo the Euler–Lagrange expressions $\delta_i(\mathcal{L})$ and their total derivatives (i.e. equality on the shell $\operatorname{Ker} \mathcal{E}_L$). From (3.7) we obtain the weak identity

$$\mathbf{L}_{J^1 u} L \approx -\, d_\lambda \mathcal{I}^\lambda\, \omega,$$

and then, expanding the Lie derivative using the first variational formula (3.5),

$$\partial_\lambda u^\lambda \mathcal{L} + \big[u^\lambda \partial_\lambda + u^i \partial_i + (d_\lambda u^i - y_\mu^i \partial_\lambda u^\mu)\partial_i^\lambda\big]\mathcal{L} \approx -d_\lambda \big[\pi_i^\lambda (u^\mu y_\mu^i - u^i) - u^\lambda \mathcal{L}\big].$$

Suppose now that the Lie derivative $\mathbf{L}_{J^1 u} L$ vanishes, i.e. the Lagrangian L is invariant under gauge transformations generated by the vector field u. Then we obtain the weak conservation law

$$0 \approx -d_\lambda [\pi_i^\lambda (u^\mu y_\mu^i - u^i) - u^\lambda \mathcal{L}]$$

for the symmetry current

$$\mathcal{I} = \mathcal{I}^\lambda \omega_\lambda, \qquad \mathcal{I}^\lambda = \pi_i^\lambda (u^\mu y_\mu^i - u^i - u^\lambda \mathcal{L}), \tag{3.8}$$

along the vector field u. Moreover, $\mathbf{L}_{J^1 u} L = 0$ implies the weak identity $d_\lambda \mathcal{I}^\lambda \approx 0$, and then pulling back along a solution s (so that $\delta_i(\mathcal{L}) \circ J^2 s = 0$) yields the differential conservation law

$$\partial_\lambda(\mathcal{I}^\lambda \circ s) = 0 \tag{3.9}$$

on solutions of the Euler-Lagrange equations (3.5). This differential conservation law implies the integral conservation law

$$\int_{\partial N} s^* \mathcal{I} = 0, \tag{3.10}$$

where N is a compact n-dimensional submanifold of X with boundary ∂N.

It may happen that a symmetry current $\mathcal{I}$ (3.8) can be put into the form

$$\mathcal{I} = W + d_H U = (W^\lambda + d_\mu U^{\mu\lambda}) \omega_\lambda. \tag{3.11}$$

where the term W contains only the variational derivatives $\delta_i \mathcal{L} = (\partial_i - d_\lambda \partial_i^\lambda)\mathcal{L}$, i.e., $W \approx 0$, and

$$U = U^{\mu\lambda} \omega_{\mu\lambda} : J^1 Y \to \Lambda^{n-2} T^* X$$

is a horizontal $(n-2)$-form on $J^1 Y \to X$. In this situation one says that $\mathcal{I}$ *reduces to the superpotential* U. Geometrically, this decomposition isolates two contributions to the current: W, which vanishes once the Euler–Lagrange equations are imposed (so it measures the failure of the field to be on-shell), and $d_H U$, which is a total divergence. Physically, the total divergence $d_H U$ means that the corresponding conserved quantity is determined entirely by a boundary term (the superpotential). Such decompositions occur in gauge theories, where Noether currents are not local densities but are rather "trivial" up to boundary contributions.

On the kernel, combining the Euler-Lagrange equations $\delta_i \mathcal{L} = 0$ yields the fundamental equation

$$\mathcal{I} - d_H U = W(\delta_i \mathcal{L}) = 0. \tag{3.12}$$

Later we will see that, in the abelian gauge theory of electromagnetism, (3.12) reproduces Maxwell's equations. If a current $\mathcal{I}$ reduces to a superpotential, then the local conservation law (3.9) and its integral version (3.10) become automatic consequences of the decomposition. At the same time, the superpotential form (3.11) yields the integral relation

$$\int_{N^{n-1}} s^* \mathcal{I} = \int_{\partial N^{n-1}} s^* U, \tag{3.13}$$

which expresses conservation in terms of flux through the boundary. This may be viewed as an integral reformulation of the Euler–Lagrange equations. Such superpotentials are a recurring feature in both gauge theory and gravitation, where symmetry generators depend on derivatives of the gauge parameters.

Now let us consider the situation with *background fields*. We will show how conservation laws can fail if these fields do not lie in the kernel (3.5). Suppose we have a fiber bundle $Y \to X$ with coordinates (x^λ, y^i) for dynamical fields, and a second bundle $Y' \to X$ with coordinates (x^λ, y^a) for background fields, which are fixed by sections

$$y^b = \phi^b(x), \quad y^b_\lambda = \partial_\lambda \phi^b(x).$$

The total configuration space is $Y_{\rm tot} = Y \times_X Y'$, and a Lagrangian L is defined on $J^1 Y_{\rm tot}$. A projectable vector field u on $Y_{\rm tot}$ projects onto Y' because gauge transformations of background fields do not depend on dynamical ones. In coordinates it has the form

$$u = u^\lambda(x^\mu)\partial_\lambda + u^a(x^\mu, y^b)\partial_a + u^i(x^\mu, y^b, y^j)\partial_i. \tag{3.14}$$

Substituting (3.14) into (3.2) yields the first variational formula in the presence of background fields,

$$\begin{aligned}\partial_\lambda u^\lambda \mathcal{L} + \big[u^\lambda\partial_\lambda + u^a\partial_a + u^i\partial_i + (d_\lambda u^a - y^a_\mu \partial_\lambda u^\mu)\partial^\lambda_a + (d_\lambda u^i - y^i_\mu\partial_\lambda u^\mu)\partial^\lambda_i\big]\mathcal{L} \\ = (u^a - y^a_\lambda u^\lambda)\partial_a\mathcal{L} + \pi^\lambda_a d_\lambda(u^a - y^a_\mu u^\mu) + (u^i - y^i_\lambda u^\lambda)\delta_i\mathcal{L} \\ - d_\lambda[\pi^\lambda_i(u^\mu y^i_\mu - u^i) - u^\lambda\mathcal{L}].\end{aligned} \tag{3.15}$$

Then by dropping the term $(u^i - y^i_\lambda u^\lambda)\delta_i\mathcal{L}$ on the right-hand side of (3.15), we get the weak identity

$$\begin{aligned}\partial_\lambda u^\lambda \mathcal{L} + \big[u^\lambda\partial_\lambda + u^a\partial_a + u^i\partial_i + (d_\lambda u^a - y^a_\mu \partial_\lambda u^\mu)\partial^\lambda_a + (d_\lambda u^i - y^i_\mu\partial_\lambda u^\mu)\partial^\lambda_i\big]\mathcal{L} \\ \approx (u^a - y^a_\lambda u^\lambda)\partial_a\mathcal{L} + \pi^\lambda_a d_\lambda(u^a - y^a_\mu u^\mu) - d_\lambda[\pi^\lambda_i(u^\mu y^i_\mu - u^i) - u^\lambda\mathcal{L}]\end{aligned}$$

which holds on the kernel (3.5). If a total Lagrangian L is invariant under gauge transformations of the product $Y_{\rm tot}$, we obtain a weak identity in the presence of background fields,

$$(u^a - y^a_\mu u^\mu)\partial_a\mathcal{L} + \pi^\lambda_a d_\lambda(u^a - y^a_\mu u^\mu) \approx d_\lambda[\pi^\lambda_i(u^\mu y^i_\mu - u^i) - u^\lambda\mathcal{L}]. \tag{3.16}$$

Thus, when background fields fail to satisfy the kernel condition (3.5), the left-hand side of (3.16) does not vanish. In other words, the would-be conserved current acquires additional source terms involving the background fields. Physically this expresses the fact that external backgrounds can inject or absorb energy-momentum or charge, and so strict conservation of the Noether current is violated.

3.3. Gauge invariance. Let $P \to X$ be a G-principal bundle. In a gauge model with symmetry group G, the gauge potentials are identified with principal connections on P, i.e., with global sections of the bundle of principal connections $C \to X$, while matter fields are represented by global sections of a P-associated vector bundle Y, called the matter bundle. Thus the total configuration space of a gauge model (with unbroken symmetries) is the product bundle

$$J^1Y_{\text{tot}} = J^1Y \times_X J^1C \to X.$$

In gauge theory, different types of gauge transformations are considered. In the most general terms, a *gauge transformation* of a principal bundle P is an automorphism Φ_P that commutes with the right G-action, i.e.,

$$R_g \circ \Phi_P = \Phi_P \circ R_g$$

for every $g \in G$. Such an automorphism of P induces a corresponding automorphism of any P-associated bundle $Y = (P \times V)/G$,

$$\begin{aligned}\Phi_Y : (P \times V)/G &\to (\Phi_P(P) \times V)/G \\ (p, v) \cdot G &\mapsto (\Phi_P(p), v) \cdot G,\end{aligned}$$

for $p \in P$, $v \in V$. Likewise, an automorphism of P determines an induced automorphism

$$\Phi_C : J^1P/G \to J^1\Phi_P(J^1P)/G \tag{3.17}$$

of the bundle $C = J^1P/G$ of principal connections.

To derive Noether conservation laws we restrict attention to vertical automorphisms of P, which we call *gauge transformations*. Every such gauge transformation is given by

$$\Phi_P(p) = pf(p), \qquad p \in P, \tag{3.18}$$

where $f : P \to G$ is a G-equivariant function satisfying

$$f(pg) = g^{-1}f(p)g, \qquad p \in P, g \in G.$$

This form amounts to the fact that vertical automorphisms act only along the fibers of P, and that any such automorphism can be described by multiplying $p \in P$ on the right by a group element that depends smoothly on p.

There is a natural one-to-one correspondence between these G-equivariant functions $f : P \to G$ and the global sections of the group bundle $P^G = (P \times G)/G$ whose typical fiber is G, acted upon by conjugation (the adjoint representation). The group bundle P^G acts fiberwise on any P-associated bundle Y by

$$\begin{aligned}P^G \times_X Y &\to Y \\ ((p, g) \cdot G, (p, v) \cdot G) &\mapsto (p, gv) \cdot G,\end{aligned}$$

for $g \in G$, $v \in V$. Thus, a gauge transformation Φ_P defined by $\Phi_P(p) = pf(p)$ corresponds to the global section $s : X \to P^G$ given by $x \mapsto (p, f(p)) \cdot G$ for any $p \in P_x$. Hence the *gauge group* $\text{Gau}(P)$ of vertical automorphisms of $P \to X$ (under composition) is canonically isomorphic to the group of global sections of the group bundle P^G.

In order to understand the structure of the gauge group, it is useful to restrict attention to certain subgroups. Here we will focus on one-parameter subgroups generated by G-invariant vertical vector fields ξ on P, called the *principal vector fields*. There is a natural one-to-one correspondence between principal vector fields on P and sections of the gauge algebra bundle $V_G P \to X$, so we may write

$$\xi = \xi^p(x) e_p, \qquad \xi \in \Gamma(V_G P),$$

where $\{e_p\}$ is a local basis of $\mathfrak{g}$ and the functions $\xi^p(x)$ are called the *gauge parameters*. The adjoint action of a principal vector field ξ_0 on another field ξ is given by the Lie bracket

$$\begin{aligned} &\xi_0 : V_G P \to V_G P \\ &\xi \mapsto [\xi_0, \xi] = c^p_{rq}\, \xi_0^r\, \xi^q\, e_p, \end{aligned}$$

where c^p_{rq} are the structure constants of $\mathfrak{g}$. In terms of gauge parameters, this leads to the transformation law

$$\xi^p \mapsto -\, c^p_{rq}\, \xi_0^r\, \xi^q. \tag{3.19}$$

by the coadjoint representation. Given a principal vector field ξ on P, there is an induced principal vector field ξ_Y on any P-associated vector bundle $Y \to X$, corresponding to the infinitesimal action of the one-parameter subgroup $\langle \Phi_Y \rangle$ of gauge transformations on Y. In local coordinates it is

$$\xi_Y = \xi^p I^i_p \partial_i,$$

where the I_p are vectors representing the Lie algebra $\mathfrak{g}$ in the chosen G-module V, i.e. the generators of the group action on the typical fiber V of Y. Concretely, if $\rho : G \to GL(V)$, then $I_p = d\rho(e_p) \in \mathfrak{gl}(V)$. Similarly, the principal vector field on the bundle of principal connections C corresponding to the infinitesimal gauge action is

$$\xi_C = \big(\partial_\mu \xi^r + c^r_{qp}\, a^q_\mu\, \xi^p\big) \partial^\mu_r. \tag{3.20}$$

Thus, the combined principal vector field on the product $C \times_X Y$ is

$$\xi_{YC} = \big(\partial_\mu \xi^r + c^r_{qp} a^q_\mu \xi^p\big) \partial^\mu_r + \xi^p I^i_p \partial_i. \tag{3.21}$$

Remark 3.2 (Collective index notation)**.** For brevity, we introduce a collective index B so that

$$u^{B\mu}_p \partial_B = \delta^r_p \partial^\mu_r, \quad u^B_p \partial_B = c^r_{qp} a^q_\mu \partial^\mu_r + I^i_p \partial_i.$$

With this notation we may rewrite

$$\xi_{YC} = (u^{B\mu}_p \partial_\mu \xi^p + u^B_p \xi^p) \partial_B.$$

◇

A Lagrangian L on the configuration space $J^1Y_{\rm tot}$ is said to be *gauge-invariant* if the identity

$$\mathbf{L}_{J^1\xi_{YC}}L = 0$$

holds for every principal vector field ξ on P. In this case, the first variational formula (3.2) yields the identity

$$0 = (u^B_p\xi^p + u^{B\mu}_p\partial_\mu\xi^p)\delta_B\mathcal{L} + d_\lambda[(u^B_p\xi^p + u^{B\mu}_p\partial_\mu\xi^p)\pi^\lambda_B] \tag{3.22}$$

where $\delta_B\mathcal{L}$ are the variational derivatives of L and the total derivative operator (1.5) is given by

$$d_\lambda = \partial_\lambda + a^p_{\lambda\mu}\partial^\mu_p + y^i_\lambda\partial_i.$$

The equation (3.22) is equivalent to the system of equations

(i) $u^B_p\delta_B\mathcal{L} + d_\mu(u^B_p\pi^\mu_B) = 0,$

(ii) $u^{B\mu}_p\delta_B\mathcal{L} + d_\lambda(u^{B\mu}_p\pi^\lambda_B) + u^B_p\pi^\mu_B = 0,$

(iii) $u^{B\lambda}_p\pi^\mu_B + u^{B\mu}_p\pi^\lambda_B = 0.$

These three conditions characterize the gauge invariance of a Lagrangian.

Let us specialize to the case of a Lagrangian

$$L : J^1C \to \Lambda^nT^*X,$$

where C is the bundle of principal connections, for free gauge fields, meaning a Lagrangian depending only on the gauge potentials a^q_μ and their first derivatives, without coupling to additional matter fields. In this case, the conditions (i)–(iii) become

(i) $c^r_{pq}\big(a^p_\mu\partial^\mu_r\mathcal{L} + a^p_{\lambda\mu}\partial^{\lambda\mu}_r\mathcal{L}\big) = 0,$

(ii) $\partial^\mu_q\mathcal{L} + c^r_{pq}a^p_\lambda\partial^{\mu\lambda}_r\mathcal{L} = 0,$

(iii) $\partial^{\mu\lambda}_p\mathcal{L} + \partial^{\lambda\mu}_p\mathcal{L} = 0.$

We utilize the canonical splitting (2.17) of the jet manifold J^1C into symmetric and antisymmetric parts:

$$a^r_{\lambda\mu} = \mathcal{S}^r_{\lambda\mu} + \tfrac{1}{2}\mathcal{F}^r_{\lambda\mu},$$

where $\mathcal{S}^r_{\lambda\mu}$ is symmetric and $\mathcal{F}^r_{\lambda\mu}$ is antisymmetric in (λ,μ). Introducing coordinates $(a^q_\mu, \mathcal{S}^r_{\mu\lambda}, \mathcal{F}^r_{\mu\lambda})$, the equations (ii) and (iii) simplify as follows. For (iii), since $\partial^{\mu\lambda}_p\mathcal{L}$ denotes differentiation with respect to $a^p_{\mu\lambda}$, decomposing into symmetric and antisymmetric parts gives

$$\partial^{\mu\lambda}_p\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\mathcal{S}^p_{\mu\lambda}} + \frac{1}{2}\frac{\partial\mathcal{L}}{\partial\mathcal{F}^p_{\mu\lambda}}.$$

Equation (iii) requires symmetry under (μ, λ), but the antisymmetric part already cancels. Therefore the condition (iii) is equivalent to

$$\frac{\partial \mathcal{L}}{\partial \mathcal{S}^r_{\mu\lambda}} = 0. \tag{3.23}$$

For (ii), we will show that this simplifies to the identity

$$\partial^\mu_q a^q_\mu = 0, \tag{3.24}$$

which expresses the dependence of $\mathcal{L}$ solely through the field strength $\mathcal{F}$.

Starting from (ii)

$$\partial^\mu_q \mathcal{L} + c^r_{pq} a^p_\lambda \partial^{\mu\lambda}_r \mathcal{L} = 0,$$

we use the canonical splitting

$$a^r_{\lambda\mu} = \mathcal{S}^r_{\lambda\mu} + \tfrac{1}{2}\mathcal{F}^r_{\lambda\mu},$$

so that differentiation with respect to $a^r_{\mu\lambda}$ decomposes as

$$\partial^{\mu\lambda}_r \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \mathcal{S}^r_{\mu\lambda}} + \frac{1}{2}\frac{\partial \mathcal{L}}{\partial \mathcal{F}^r_{\mu\lambda}}.$$

By (iii) we have $\dfrac{\partial \mathcal{L}}{\partial \mathcal{S}^r_{\mu\lambda}} = 0$, hence

$$\partial^{\mu\lambda}_r \mathcal{L} = \frac{1}{2}\frac{\partial \mathcal{L}}{\partial \mathcal{F}^r_{\mu\lambda}}.$$

Substituting into (ii) gives

$$\partial^\mu_q \mathcal{L} + \tfrac{1}{2} c^r_{pq} a^p_\lambda \frac{\partial \mathcal{L}}{\partial \mathcal{F}^r_{\mu\lambda}} = 0. \tag{3.25}$$

Now using

$$\mathcal{F}^r_{\alpha\beta} = a^r_{\alpha\beta} - a^r_{\beta\alpha} + c^r_{st} a^s_\alpha a^t_\beta,$$

we find

$$\frac{\partial \mathcal{F}^r_{\alpha\beta}}{\partial a^q_\mu} = c^r_{qt}\, \delta^\mu_\alpha\, a^t_\beta + c^r_{sq}\, a^s_\alpha\, \delta^\mu_\beta.$$

Hence the chain rule yields

$$\partial^\mu_q \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \mathcal{F}^r_{\alpha\beta}} \frac{\partial \mathcal{F}^r_{\alpha\beta}}{\partial a^q_\mu} = c^r_{qt}\, a^t_\beta\, \frac{\partial \mathcal{L}}{\partial \mathcal{F}^r_{\mu\beta}} + c^r_{sq}\, a^s_\alpha\, \frac{\partial \mathcal{L}}{\partial \mathcal{F}^r_{\alpha\mu}}.$$

In the second sum rename $\alpha \leftrightarrow \beta$ and $s \leftrightarrow t$; then use the antisymmetry $\dfrac{\partial \mathcal{L}}{\partial \mathcal{F}^r_{\alpha\mu}} = -\dfrac{\partial \mathcal{L}}{\partial \mathcal{F}^r_{\mu\alpha}}$

to obtain

$$\partial_q^\mu \mathcal{L} = \left(c_{qt}^r - c_{tq}^r\right) a_\beta^t \,\frac{\partial \mathcal{L}}{\partial \mathcal{F}_{\mu\beta}^r}.$$

Since the structure constants are antisymmetric in the lower indices, $c_{qt}^r = -c_{tq}^r$, this simplifies to

$$\partial_q^\mu \mathcal{L} = 2c_{qt}^r \, a_\beta^t \,\frac{\partial \mathcal{L}}{\partial \mathcal{F}_{\mu\beta}^r}.$$

Combining this with (3.25) yields (3.24). In essence, (iii) kills the symmetric dependence on second jets, and (ii) then forces any remaining first-order dependence of $\mathcal{L}$ on the variables a_μ^q to appear only through the antisymmetric combination $\mathcal{F}_{\lambda\mu}^r$. This is the algebraic underpinning of the heuristic "gauge invariance implies dependence only on the curvature."

A glance at the equations (3.23) and (3.24) shows that the gauge-invariant Lagrangian $L : J^1C \to \Lambda^n T^*X$ factorizes through the field strength $\mathcal{F}$ of gauge potentials, i.e.,

$$L = \overline{L} \circ \mathcal{F} : J^1C \to C_- \to \Lambda^n T^*X. \tag{3.26}$$

Using this, the equation (i) can be written as

$$c_{pq}^r \mathcal{F}_{\lambda\mu}^p \frac{\partial \mathcal{L}}{\partial \mathcal{F}_{\lambda\mu}^r} = 0,$$

which is an equivalent way of formulating the gauge invariance of the Lagrangian L.

3.4. Yang-Mills Lagrangian. We discuss the *Yang-Mills Lagrangian* L_{YM} of gauge potentials on the configuration space J^1C in the presence of a background metric g on the base X. It is given by

$$L_{YM} = \frac{1}{4\varepsilon^2} a_{pq}^G g^{\lambda\mu} g^{\beta\nu} \mathcal{F}_{\lambda\beta}^p \mathcal{F}_{\mu\nu}^q \sqrt{|\det(g_{\mu\nu})|}\, \omega, \tag{3.27}$$

where a^G is a nondegenerate G-invariant metric in the Lie algebra $\mathfrak{g}_r$ and ε is a coupling constant. The equations (i)-(iii) with the Lagrangian $L = L_{YM}$ are called the *Yang-Mills equations*, and a principal connection A solving the equations is called a *Yang-Mills connection*. The Yang-Mills theory has become widespread since the foundational work of Atiyah, Donaldson, Witten, and many others, cf. [AJ78, W94, T94, DK97, CVB03].

We note the following useful fact that arises in relation with the Yang-Mills Lagrangian (3.27). If one chooses an affine connection $\Gamma : C \to J^1C$ on the bundle of principal connections $C \to X$, then the identity (2.18) shows that the Yang-Mills Lagrangian L_{YM} factorizes through the covariant differential associated with the connection $\mathcal{S} \circ \Gamma$ on $C \to X$, where $\mathcal{S} = \mathrm{pr}_1 : J^1C \to C_+$.

On the kernel of the Euler–Lagrange operator $\mathcal{E}_L$ (3.5), the identity (3.22) becomes the weak conservation law

$$0 \approx d_\lambda\big[(u_p^B \xi^p + u_p^{B\mu} \partial_\mu \xi^p)\pi_B^\lambda\big] \tag{3.28}$$

of the Noether current

$$\mathcal{I}^\lambda = -(u_p^B \xi^p + u_p^{B\mu} \partial_\mu \xi^p) \pi_B^\lambda. \tag{3.29}$$

Accordingly, the equalities (i)–(iii) on the kernel (3.5) lead to the familiar Noether identities for a gauge-invariant Lagrangian L:

(i) $d_\mu(u_p^B \pi_B^\mu) \approx 0.$

(ii) $d_\lambda(u_p^{B\mu} \pi_B^\lambda) + u_p^B \pi_B^\mu \approx 0.$

(iii) $u_p^{B\lambda} \pi_B^\mu + u_p^{B\mu} \pi_B^\lambda = 0.$

This system is equivalent to the weak equality (3.28) because the latter must hold for *arbitrary* gauge parameters $\xi^p(x)$. Expanding (3.28) in powers of ξ and its derivatives, the coefficients of ξ^p, $\partial_\mu \xi^p$, and $\partial_\mu \partial_\lambda \xi^p$ must each vanish separately, which yields precisely the three identities (i)–(iii).

A glance at (3.28) and (3.29) shows that the Noether conservation law and current are written explicitly in terms of gauge parameters. The weak identities (i)–(iii) ensure that this dependence is compatible with gauge covariance. Concretely, they guarantee that if the conservation law holds for a choice of ξ, then it also holds after an arbitrary variation $\xi \mapsto \xi + \delta\xi$. In this sense the conservation law is gauge-covariant, it remains consistent when the parameters are changed by the coadjoint representation (3.19). Thus, the parameter dependence of the current is the mechanism that enforces gauge invariance of the conservation law.

The equations (i)–(iii) are not mutually independent, in fact (i) follows from (ii) and (iii). This redundancy reflects the fact that the current (3.29) can be rewritten in *superpotential form*

$$\mathcal{I}^\lambda = \xi^p u_p^{B\lambda} \delta_B \mathcal{L} - d_\mu(\xi^p u_p^{B\mu} \pi_B^\lambda), \tag{3.30}$$

where the antisymmetric superpotential is

$$U^{\mu\lambda} = -\,\xi^p u_p^{B\mu} \pi_B^\lambda.$$

Since a matter field Lagrangian does not involve the second-order jet coordinates, the expression of $U^{\mu\lambda}$ simplifies and the *Noether superpotential* reduces to

$$U^{\mu\lambda} = \xi^p \pi_p^{\mu\lambda}, \tag{3.31}$$

so that it depends only on the gauge potentials and their first derivatives, and not on matter fields.

The corresponding integral relation (3.13) reads

$$\int_{N^{n-1}} s^* \mathcal{I}^\lambda \, \omega_\lambda = \int_{\partial N^{n-1}} s^* \big(\xi^p \pi_p^{\mu\lambda} \big) \, \omega_{\mu\lambda}, \tag{3.32}$$

where N^{n-1} is a compact oriented $(n-1)$-dimensional submanifold of X with boundary ∂N^{n-1}. This expresses the current–field relation in integral form: the flux of the Noether current through N^{n-1} is entirely determined by the superpotential flux across ∂N^{n-1}. In physical terms, one can view (3.32) as relating a symmetry current to the gauge field sourced by it. In the abelian case of electromagnetism, the analogous relation reduces to the familiar balance between the electric current and the electromagnetic field it generates. The key difference is that in the abelian case the gauge parameter dependence drops out, while in the nonabelian case the current retains explicit ξ-dependence as a manifestation of gauge covariance.

Example 3.3 (Abelian gauge model)**.** As we saw above, in the nonabelian case the Noether current and the conservation law generally depend on the gauge parameters $\xi^p(x)$, and we showed that this dependence is controlled by the Noether identities (i)–(iii), which guarantee gauge covariance. By contrast, for an abelian symmetry group G the situation simplifies, and one can take the Noether current and conservation law to be independent of gauge parameters.

Let us consider the electromagnetic theory, where $G = U(1)$ and the infinitesimal action on the fiber is $y \mapsto iy$. In this case, a gauge parameter ξ is not transformed under the coadjoint action (3.19), so it may be chosen as a constant. Setting $\xi = 1$ for convenience, the Noether current (3.29) becomes

$$\mathcal{I}^\lambda = -u^B \pi^\lambda_B.$$

For $U(1)$, this reduces further to

$$\mathcal{I}^\lambda = -iy^j \pi^\lambda_j.$$

Thus in the abelian case the Noether current is independent of gauge potentials and remains invariant under gauge transformations. Physically, this current is (up to sign) the familiar electric current carried by matter fields.

In this case the weak conservation law for the Noether current (3.28) reduces to the continuity equation

$$d_\lambda \mathcal{I}^\lambda \approx 0,$$

and the integral relation (3.10) becomes

$$\int_{\partial N} s^*(y^j \pi^\lambda_j)\omega_\lambda = 0,$$

for any compact n-dimensional submanifold $N \subset X$ with boundary ∂N. This is precisely the integral equation of continuity, expressing charge conservation: the total flux of the electric current through ∂N vanishes.

When $\xi = 1$, the electromagnetic superpotential (3.31) takes the form

$$U^{\mu\lambda} = \pi^{\mu\lambda} = -\frac{1}{4\pi}\,\mathcal{F}^{\mu\lambda},$$

where $\mathcal{F}$ is the electromagnetic field strength. Substituting this into (3.12) yields

$$\frac{1}{4\pi}d_\mu \mathcal{F}^{\mu\lambda} = iy^j \pi_j^\lambda,$$

which is exactly the system of inhomogeneous Maxwell equations: the divergence of the electromagnetic field strength equals the electric current density. Accordingly, the integral relation (3.32) is the integral form of Maxwell's equations. In particular, one recovers Gauss's law: the flux of the electric field through a closed surface is equal to the total electric charge contained inside the surface. ◇

Let us now turn to energy-momentum conservation laws in gauge theory. For the sake of simplicity, we will consider only gauge theory without matter fields. We work with the Yang-Mills Lagrangian L_{YM} on the jet manifold J^1C. First recall the construction from Example 2.4. Given a vector field τ on X, let A be a principal connection on the principal bundle $P \to X$, and let

$$\tau_A = \tau^\lambda(\partial_\lambda + A_\lambda^p \epsilon_p)$$

denote the horizontal lift of τ onto P by means of the connection A. This vector field, in turn, gives rise to the vector field $\widetilde{\tau}_A$ on the bundle of principal connections C, namely

$$\widetilde{\tau}_A = \tau^\lambda \partial_\lambda + [\tau^\lambda(\partial_\mu A_\lambda^r + c_{pq}^r a_\mu^p A_\lambda^q) - \partial_\mu \tau^\beta (a_\beta^r - A_\beta^r)]\partial_r^\mu. \tag{3.33}$$

We will now derive the energy-momentum current along the vector field $\widetilde{\tau}_B$.

Since the Yang-Mills Lagrangian L_{YM} also depends on a background metric, we will consider the total Lagrangian

$$L_{YM} = \frac{1}{4\varepsilon^2} a_{pq}^G \sigma^{\lambda\mu}\sigma^{\beta\nu}\mathcal{F}_{\lambda\beta}^p \mathcal{F}_{\mu\nu}^q \sqrt{|\det(\sigma_{\mu\nu})|}\omega, \tag{3.34}$$

with respect to a metric σ, on the total configuration space $J^1(C \times_X \operatorname{Sym}^2 TX)$, where the tensor bundle $\operatorname{Sym}^2 TX$ is provided with the holonomic coordinates $(x^\lambda, \sigma^{\mu\nu})$. Given a vector field τ on X, it has a canonical lift

$$\widetilde{\tau} = \tau^\lambda \partial_\lambda + (\partial_\nu \tau^\alpha \sigma^{\nu\beta} + \partial_\nu \tau^\beta \sigma^{\nu\alpha})\partial_{\alpha\beta}$$

onto the tensor bundle $\operatorname{Sym}^2 T^*X$, which is the generator of a local 1-parameter group of general covariant transformations of $\operatorname{Sym}^2 T^*X$. Thus, we have a lift

$$\begin{aligned}\widetilde{\tau}_A = \tau^\lambda \partial_\lambda &+ \left[\tau^\lambda(\partial_\mu A_\lambda^r + c_{pq}^r a_\mu^p A_\lambda^q) - \partial_\mu \tau^\beta (a_\beta^r - A_\beta^r)\right]\partial_r^\mu \\ &+ (\partial_\nu \tau^\alpha \sigma^{\nu\beta} + \partial_\nu \tau^\beta \sigma^{\nu\alpha})\partial_{\alpha\beta}\end{aligned} \tag{3.35}$$

of a vector field τ on X onto the product $C \times_X \operatorname{Sym}^2 T^*X$. For the sake of simplicity, we denote it by the same symbol $\widetilde{\tau}_A$.

Our next task is to derive an explicit formula for the Noether current in the case of the

Yang-Mills Lagrangian $L = L_{YM}$ and Yang-Mills connection A, i.e. the energy-moment current $\mathcal{I}_A$ along the vector field $\widetilde{\tau}_A$. We start from the first variational formula. For a projectable vector field v on the total configuration bundle, the first variational formula (3.2) reads

$$\mathbf{L}_{J^1 v}\, L = (\delta\phi^B)\,\delta_B\mathcal{L} \;+\; d_\lambda\big(\pi^\lambda_B\,\delta\phi^B - v^\lambda\mathcal{L}\big),$$

where ϕ^B denotes all fields, $\delta_B\mathcal{L}$ are the Euler–Lagrange expressions, $\pi^\lambda_B = \partial\mathcal{L}/\partial(\partial_\lambda\phi^B)$ are the canonical momenta, and $\delta\phi^B$ is the induced field variation.

Specializing to the symmetry generated by τ, we take v to be the lift $\widetilde{\tau}_A$ of the base vector field τ defined in (3.35). Then $\mathbf{L}_{J^1\widetilde{\tau}_A} L = 0$ because the Lagrangian is invariant under the base diffeomorphism and gauge transformation generated by $\widetilde{\tau}_A$. Thus

$$0 = (\delta\phi^B)\,\delta_B\mathcal{L} + d_\lambda\big(\pi^\lambda_B\,\delta\phi^B - \tau^\lambda\mathcal{L}\big).$$

Now restrict to the Yang–Mills kernel. For pure Yang–Mills (no matter fields) the only dynamical fields ϕ^B are the gauge potentials a^r_ν (and we treat the metric as a fixed background for this step). Hence $\pi^\lambda_B\delta\phi^B$ reduces to $\pi^{\lambda\nu}_r\delta a^r_\nu$, with $\delta a^r_\nu = \mathbf{L}_\tau A^r_\nu$ the induced variation of the connection components. On the Yang-Mills kernel we have $\delta_B\mathcal{L} \approx 0$, so the bulk term vanishes and we are left with the weak conservation law

$$d_\lambda\big(\pi^{\lambda\nu}_r\delta a^r_\nu - \tau^\lambda\mathcal{L}\big) \approx 0.$$

Recalling the definition (3.29), the Noether current associated to $\widetilde{\tau}_A$ is the horizontal $(n-1)$-form whose components are the quantity inside the divergence. Thus the *energy-momentum current* is defined by

$$\mathcal{I}^\lambda_A \;=\; \pi^{\lambda\nu}_r\,\delta a^r_\nu \;-\; \tau^\lambda\mathcal{L}. \tag{3.36}$$

Going a step further, if one inserts the explicit expression for δa^r_ν given by the lifted field variation (the bracketed expression in (3.35)), and expands $\pi^{\lambda\nu}_r\delta a^r_\nu$, one obtains

$$\mathcal{I}^\lambda_A = \pi^{\lambda\nu}_r\big[-\tau^\mu(\partial_\nu A^r_\mu + c^r_{pq}a^p_\nu A^q_\mu - a^r_{\mu\nu}) + \partial_\nu\tau^\mu(a^r_\mu - A^r_\mu)\big] - \tau^\lambda\mathcal{L}. \tag{3.37}$$

The total Lagrangian (3.34) is by construction invariant under gauge transformations and general covariant transformations. Hence its Lie derivative along the vector field $\widetilde{\tau}_A$ vanishes. Using the general formula (3.16) for weak identities in the presence of a background field, one obtains, on the Yang-Mills kernel,

$$0 \approx (\partial_\nu\tau^\alpha g^{\nu\beta} + \partial_\nu\tau^\beta g^{\nu\alpha} - \partial_\lambda g^{\alpha\beta}\tau^\lambda)\partial_{\alpha\beta}\mathcal{L} - d_\lambda\mathcal{I}^\lambda_A. \tag{3.38}$$

The weak identity (3.38) can be rewritten as

$$0 \approx \partial_\lambda\tau^\mu t^\lambda_\mu\sqrt{|\det(g)|} - \tau^\mu C^\beta_{\mu\lambda}t^\lambda_\beta\sqrt{|\det(g)|} - d_\lambda\mathcal{I}^\lambda_A \tag{3.39}$$

where $C^\beta_{\mu\lambda}$ are the Christoffel symbols of the Levi-Civita connection for g and t^μ_β are the components of the metric energy-momentum tensor of the gauge field, which is defined by the formula

$$\delta_g L_{YM} \;=\; \frac{1}{2}\sqrt{|g|}\, t^{\mu\nu}\, \delta g_{\mu\nu}. \tag{3.40}$$

Remark 3.4 (Metric energy-momentum tensor)**.** Regarding our definition of the metric energy-momentum tensor via (3.40). Treating L_{YM} locally as a function of the metric components $g_{\alpha\beta}$, we have

$$\delta_g L_{YM} \;=\; \frac{\partial L_{YM}}{\partial g_{\alpha\beta}}\, \delta g_{\alpha\beta}.$$

Comparing with (3.40) and equating coefficients of the variations $\delta g_{\alpha\beta}$ gives

$$\frac{\partial L_{YM}}{\partial g_{\alpha\beta}} \;=\; \frac{1}{2}\sqrt{|g|}\, t^{\alpha\beta}.$$

Multiplying both sides by $g^{\mu\alpha}$ yields

$$g^{\mu\alpha}\frac{\partial L_{YM}}{\partial g_{\alpha\beta}} = \frac{1}{2}\sqrt{|g|}\, g^{\mu\alpha}t^{\alpha\beta} = \frac{1}{2}\sqrt{|g|}\, t^{\mu}{}_{\beta},$$

and therefore

$$t^{\mu}{}_{\beta}\sqrt{|\det(g)|} = 2g^{\mu\alpha}\partial_{\alpha\beta}L_{YM}$$

which is another common definition of the metric energy-momentum tensor.

If instead one regards the independent metric variables to be the inverse metric $g^{\alpha\beta}$ (i.e. choosing the coordinates $g^{\alpha\beta}$ on $\mathrm{Sym}^2 TX$), we may write the metric variation of L_{YM} as

$$\delta_g L_{YM} \;=\; \frac{\partial L_{YM}}{\partial g^{\alpha\beta}}\, \delta g^{\alpha\beta}.$$

Then using the relation $\delta g^{\alpha\beta} = -g^{\alpha\mu}g^{\beta\nu}\delta g_{\mu\nu}$ and comparing with (3.40) gives

$$\frac{\partial L_{YM}}{\partial g^{\alpha\beta}}(-g^{\alpha\mu}g^{\beta\nu}\delta g_{\mu\nu}) = \frac{1}{2}\sqrt{|g|}\, t^{\mu\nu}\delta g_{\mu\nu}.$$

Equating coefficients of $\delta g_{\mu\nu}$ yields

$$-\, g^{\alpha\mu}g^{\beta\nu}\frac{\partial L_{YM}}{\partial g^{\alpha\beta}} = \frac{1}{2}\sqrt{|g|}\, t^{\mu\nu}.$$

Multiplying by$g_{\nu\beta}$ lowers an index so that

$$-\,\frac{\partial L_{YM}}{\partial g^{\alpha\beta}}\, g^{\mu\alpha} = \frac{1}{2}\sqrt{|g|}\, t^{\mu}{}_{\beta}.$$

Hence

$$t^{\mu}{}_{\beta}\sqrt{|g|} \;=\; -\,2\, g^{\mu\alpha}\, \frac{\partial L_{YM}}{\partial g^{\alpha\beta}}$$

so a minus sign appears if $\partial/\partial g^{\alpha\beta}$ is used instead of $\partial/\partial g_{\alpha\beta}$. As always, one should be careful with conventions. ◇

In the presence of a background field g we will use the notation

$$\sqrt{|g|} = \sqrt{\det |g|}\,\omega, \quad \omega = dx^1 \wedge \cdots \wedge dx^n$$

to denote the Riemannian density.

Fact 3.5. *For the Yang-Mills Lagrangian $L_{YM} = \mathcal{L}\sqrt{|g|}$ with background field g, we have the following identity for the momentum conjugate to $\partial_\lambda A^r_\nu$:*

$$\pi_r^{\lambda\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\lambda A^r_\nu)} = \frac{1}{\varepsilon^2}\, a^G_{rq}\, \mathcal{F}^{q\,\lambda\nu}\sqrt{|g|} \tag{3.41}$$

where $F^{q\,\lambda\mu} = g^{\lambda\alpha} g^{\mu\beta} F^q_{\alpha\beta}$.

Proof. For the Yang–Mills Lagrangian (3.34) we use the fact (3.26) that L_{YM} factors through $\mathcal{F}$, together with the chain rule. Indeed, the Yang–Mills Lagrangian depends on the connection a^r_μ only through the field strength

$$\mathcal{F}^r_{\mu\nu} = \partial_\mu a^r_\nu - \partial_\nu a^r_\mu + c^r_{pq} a^p_\mu a^q_\nu.$$

By the chain rule, the momentum conjugate to a^r_μ is

$$\pi_r^{\lambda\mu} = \frac{\partial \mathcal{L}}{\partial(\partial_\lambda a^r_\mu)} = \frac{\partial \mathcal{L}}{\partial \mathcal{F}^q_{\alpha\beta}} \frac{\partial \mathcal{F}^q_{\alpha\beta}}{\partial(\partial_\lambda a^r_\mu)}.$$

Next, using the linear dependence of $\mathcal{F}$ on the derivatives of a,

$$\frac{\partial \mathcal{F}^q_{\alpha\beta}}{\partial(\partial_\lambda a^r_\mu)} = \delta^q_r(\delta^\lambda_\alpha \delta^\mu_\beta - \delta^\lambda_\beta \delta^\mu_\alpha),$$

which reflects the antisymmetry $\mathcal{F}^q_{\alpha\beta} = -\mathcal{F}^q_{\beta\alpha}$. Therefore,

$$\pi_r^{\lambda\mu} = \frac{\partial \mathcal{L}}{\partial \mathcal{F}^q_{\alpha\beta}}\, \delta^q_r(\delta^\lambda_\alpha \delta^\mu_\beta - \delta^\lambda_\beta \delta^\mu_\alpha) = 2\,\frac{\partial \mathcal{L}}{\partial \mathcal{F}^r_{\lambda\mu}},$$

where the factor of 2 arises from the antisymmetry of $\mathcal{F}^r_{\alpha\beta}$. Finally, inserting the explicit form of the Lagrangian L_{YM} (3.27) yields the formula (3.41). ■

Fact 3.6. *For the metric energy-momentum tensor, the following identity holds:*

$$t^\lambda_\mu \sqrt{|g|} = \pi_q^{\lambda\nu} \mathcal{F}^q_{\mu\nu} - \delta^\lambda_\mu L_{YM}. \tag{3.42}$$

Proof. We use the density notation $\sqrt{|g|}$ and set $t^\mu{}_\nu = g^{\mu\alpha} t_{\alpha\nu}$. The Yang–Mills Lagrangian density is

$$L_{YM} = \mathcal{L}\sqrt{|g|} \qquad \mathcal{L} \;=\; \frac{1}{4\varepsilon^2}\, a^G_{pq}\, \mathcal{F}^p_{\alpha\beta} \mathcal{F}^{q\,\alpha\beta}.$$

Varying L_{YM} with respect to the metric $g_{\mu\nu}$ yields two contributions: the variation of the volume factor $\sqrt{|g|}$ and the variation of $\mathcal{F}^{q\,\alpha\beta}$ through the metric used to raise indices. Using $\delta\sqrt{|g|} = \frac{1}{2}\sqrt{|g|}g^{\mu\nu}\delta g_{\mu\nu}$, we obtain

$$\begin{aligned}\delta L_{YM} &= \frac{1}{4\varepsilon^2} a^G_{pq}\Big[(\delta\sqrt{|g|})\,\mathcal{F}^p_{\alpha\beta}\mathcal{F}^{q\,\alpha\beta} + \sqrt{|g|}\,\delta(F^p_{\alpha\beta}\mathcal{F}^{q\,\alpha\beta})\Big] \\ &= \frac{1}{4\varepsilon^2} a^G_{pq}\sqrt{|g|}\Big[\tfrac{1}{2}g^{\mu\nu}\mathcal{F}^p_{\alpha\beta}\mathcal{F}^{q\,\alpha\beta}\,\delta g_{\mu\nu} + \delta(F^p_{\alpha\beta}\mathcal{F}^{q\,\alpha\beta})\Big].\end{aligned}$$

Now vary the contraction $\mathcal{F}^p_{\alpha\beta}\mathcal{F}^{q\,\alpha\beta}$ by varying one raised index,

$$\delta(\mathcal{F}^p_{\alpha\beta}\mathcal{F}^{q\,\alpha\beta}) = -2\,\mathcal{F}^{q\,\alpha}{}_{\gamma}\mathcal{F}^{p\,\beta\gamma}\,\delta g_{\alpha\beta}.$$

and hence

$$\delta L_{YM} = \frac{1}{4\varepsilon^2} a^G_{pq}\sqrt{|g|}\Big[\tfrac{1}{2}g^{\mu\nu}\mathcal{F}^p_{\alpha\beta}\mathcal{F}^{q\,\alpha\beta} - 2\,\mathcal{F}^{q\,\mu}{}_{\rho}\mathcal{F}^{p\,\nu\rho}\Big]\delta g_{\mu\nu}.$$

Comparing with $\delta L_{YM} = \frac{1}{2}t^{\mu\nu}\sqrt{|g|}\,\delta g_{\mu\nu}$ from the definition (3.40) yields

$$t^{\mu\nu} = \frac{1}{2\varepsilon^2} a^G_{pq}\Big[\tfrac{1}{2}g^{\mu\nu}\mathcal{F}^p_{\alpha\beta}\mathcal{F}^{q\,\alpha\beta} - 2\,\mathcal{F}^{q\,\mu}{}_{\rho}\mathcal{F}^{p\,\nu\rho}\Big].$$

Lowering/raising indices and rearranging gives the equivalent form

$$t_{\mu}{}^{\lambda}\sqrt{|g|} = \frac{1}{\varepsilon^2} a^G_{pq}\,\mathcal{F}^{p\,\lambda\nu}\mathcal{F}^q_{\mu\nu}\sqrt{|g|} \;-\; \delta^\lambda_\mu L_{YM}.$$

Finally, recall the identity (3.41) for the conjugate momenta $\pi^{\lambda\nu}_r$. Substituting (3.41) into the previous expression yields

$$t^\lambda_\mu\sqrt{|g|} \;=\; \pi^{\lambda\nu}_q\mathcal{F}^q_{\mu\nu} \;-\; \delta^\lambda_\mu L_{YM},$$

which is the identity to be shown. ■

In particular, suppose $\mathbb{A}$ is a solution of the Yang-Mills equations. Taking the lift $\widetilde{\tau}_{\mathbb{A}}$ (3.35) with $A = \mathbb{A}$, the energy-momentum current (3.37) reduces to

$$\mathcal{I}^\lambda_{\mathbb{A}} \circ \mathbb{A} = \tau^\mu(t^\lambda_\mu \circ \mathbb{A})\sqrt{|\det(g)|}.$$

Then the weak identity (3.39) on the Yang-Mills connection $\mathbb{A}$ becomes

$$0 \approx -C^\beta_{\mu\lambda}(t^\lambda_\beta \circ \mathbb{A})\sqrt{|\det(g)|} - d_\lambda\left((t^\lambda_\mu \circ \mathbb{A})\sqrt{|\det(g)|}\right).$$

This is exactly the familiar covariant conservation law

$$\nabla_{\partial_\lambda}\left((t^\lambda_\mu \circ \mathbb{A})\sqrt{|\det(g)|}\right) = 0, \tag{3.43}$$

where $\nabla_{\partial_\lambda}$ is the Levi-Civita covariant derivative for the background metric g.

Note that, in the case of an arbitrary principal connection A, the corresponding weak identity (3.39) differs from (3.43) by the Noether conservation law

$$0 \approx d_\lambda(\xi^r_\nu \pi^{\lambda\nu}_r), \tag{3.44}$$

where

$$\xi_C = \xi^r_\nu \partial^\nu_r = (\partial_\nu \xi^r + c^r_{qp} a^q_\nu \xi^p)\partial^\nu_r, \quad \xi^r = \tau^\mu (A^r_\mu - \mathbb{A}^r_\mu)$$

is the principal vector field (3.20) on C. This observation leads to the following key idea: conservation laws for two different principal connections A and $\mathbb{A}$ differ only by a Noether current, which reduces to a superpotential. Thus, while the explicit expression for the current depends on the choice of connection, the covariant conservation law (3.43) remains the physically relevant content.

3.5. Belinfante–Rosenfeld superpotential. It is instructive to compare the above discussion with the familiar distinction between the *canonical* and the *metric* stress-energy tensors in classical field theory. Given a Lagrangian L with background metric g, Noether's theorem applied to translations yields the stress-energy tensor

$$T^{\lambda\,\mathrm{can}}_{\ \mu} = \frac{\partial L}{\partial(\partial_\lambda \phi^a)}\, \partial_\mu \phi^a - \delta^\lambda_\mu L,$$

where ϕ^a denotes generic field variables. This tensor is conserved in the weak sense, but in general it is neither symmetric nor gauge-invariant. On the other hand, varying the action with respect to the metric g produces the *metric* stress-energy tensor

$$T^{\lambda\,\mathrm{met}}_{\ \mu} \sqrt{|\det g|} = 2 g^{\lambda\nu}\, \partial_{\nu\mu} L,$$

which is symmetric by construction and covariantly conserved with respect to the Levi-Civita connection:

$$\nabla_{\partial_\lambda} \left(T^{\lambda\,\mathrm{met}}_{\ \mu} \sqrt{|\det g|} \right) = 0.$$

We recall the Belinfante–Rosenfeld procedure in classical field theory. In general, the canonical tensor T^{can} can be made symmetric and gauge-invariant by adding the divergence of an antisymmetric tensor, the *Belinfante–Rosenfeld superpotential*:

$$T^{\lambda\,\mathrm{BR}}_{\ \mu} = T^{\lambda\,\mathrm{can}}_{\ \mu} + \partial_\nu U^{\nu\lambda}{}_\mu, \qquad U^{\nu\lambda}{}_\mu = -U^{\lambda\nu}{}_\mu.$$

This improved tensor T^{BR} is exactly the metric stress–energy tensor obtained by varying the action with respect to g.

Fact 3.7. *$L = L_{YM}$ be the Yang-Mills Lagrangian and let A be a Yang-Mills connection. For the Noether current $\mathcal{I}^\lambda_A$ we have the on-shell decomposition*

$$\mathcal{I}^\lambda_A \;\approx\; \tau^\mu t^{\ \lambda}_\mu \sqrt{|g|} \;+\; d_\nu U^{\nu\lambda}$$

in terms of the metric energy-momentum tensor t^λ_μ and the Noether superpotential $U^{\nu\lambda}$.

Proof. We work with the Lie-algebra-valued forms

$$A = A^r_\mu \epsilon_r \, dx^\mu, \qquad \mathcal{F} = \tfrac{1}{2}\mathcal{F}^r_{\mu\nu}\epsilon_r \, dx^\mu \wedge dx^\nu,$$

and let D denote the covariant derivative $D = d + [A, \cdot]$. Cartan's formula for the Lie derivative of the connection along the base vector field $\tau = \tau^\mu \partial_\mu$ yields

$$\mathbf{L}_\tau A \; = \; \iota_\tau \mathcal{F} \; + \; D(\iota_\tau A).$$

In components this reads

$$\mathbf{L}_\tau A^r_\nu \; = \; \tau^\mu \mathcal{F}^r_{\mu\nu} \; + \; D_\nu(\tau^\mu A_\mu)^r, \tag{3.45}$$

where

$$D_\nu(\tau^\mu A_\mu)^r = \partial_\nu(\tau^\mu A^r_\mu) + c^r_{pq} a^p_\nu (\tau^\mu A^q_\mu).$$

Recall from (3.36) that the Noether current associated to the diffeomorphism generated by τ is given by

$$\mathcal{I}^\lambda_A \; = \; \pi^{\lambda\nu}_r \, \delta a^r_\nu \; - \; \tau^\lambda \mathcal{L},$$

where $\pi^{\lambda\nu}_r = \partial\mathcal{L}/\partial(\partial_\lambda a^r_\nu)$ and $\delta a^r_\nu = \mathbf{L}_\tau A^r_\nu$ is the field variation induced by τ. Using (3.45) and setting $\zeta^r := \tau^\mu A^r_\mu$, we obtain

$$\begin{aligned} \mathcal{I}^\lambda_A &= \pi^{\lambda\nu}_r \big(\tau^\mu \mathcal{F}^r_{\mu\nu} + D_\nu \zeta^r\big) - \tau^\lambda \mathcal{L} \\ &= \tau^\mu \, \pi^{\lambda\nu}_r \mathcal{F}^r_{\mu\nu} \; + \; \pi^{\lambda\nu}_r D_\nu \zeta^r \; - \; \tau^\lambda \mathcal{L}. \end{aligned} \tag{3.46}$$

Now rewrite the second term by the covariant product rule:

$$\pi^{\lambda\nu}_r D_\nu \zeta^r \; = \; D_\nu(\pi^{\lambda\nu}_r \zeta^r) \; - \; (D_\nu \pi^{\lambda\nu}_r) \, \zeta^r.$$

Hence (3.46) becomes

$$\mathcal{I}^\lambda_A = \tau^\mu \big(\pi^{\lambda\nu}_r \mathcal{F}^r_{\mu\nu} - \delta^\lambda_\mu \mathcal{L}\big) \; + \; D_\nu(\pi^{\lambda\nu}_r \zeta^r) \; - \; (D_\nu \pi^{\lambda\nu}_r) \, \zeta^r. \tag{3.47}$$

Moreover, the weak vacuum form of the Yang-Mills Euler-Lagrange equations imply that the covariant divergence of π vanishes on-shell, i.e.

$$D_\nu \pi^{\nu\lambda}_r \; \approx \; 0 \quad \text{or equivalently} \quad D_\nu(\partial\mathcal{L}/\partial\mathcal{F}^r_{\nu\lambda}) \approx 0.$$

Thus the last term in (3.47) vanishes on-shell, and we obtain the on-shell identity

$$\mathcal{I}^\lambda_A \; \approx \; \tau^\mu \big(\pi^{\lambda\nu}_r \mathcal{F}^r_{\mu\nu} - \delta^\lambda_\mu \mathcal{L}\big) \; + \; D_\nu(\pi^{\lambda\nu}_r \zeta^r).$$

Finally note that $D_\nu(\pi^{\lambda\nu}_r \zeta^r)$ equals the ordinary divergence $d_\nu(\pi^{\lambda\nu}_r \zeta^r)$ up to connection terms which are already accounted for by the on-shell vanishing of $D_\nu \pi^{\nu\lambda}_r$. Setting $U^{\nu\lambda} = \pi^{\nu\lambda}_r \zeta^r = \pi^{\nu\lambda}_r(\tau^\mu A^r_\mu)$, and noting that $\pi^{\nu\lambda}_r$ is antisymmetric in ν, λ, we may rewrite

$D_\nu(\pi_r^{\lambda\nu}\zeta^r) = d_\nu U^{\nu\lambda}$. Moreover, using the identity (3.42) for the metric energy-momentum tensor

$$t_\mu^{\ \lambda}\sqrt{|g|} \ = \ \pi_r^{\lambda\nu}\mathcal{F}_{\mu\nu}^r - \delta_\mu^\lambda \mathcal{L}.$$

we arrive at

$$\mathcal{I}_A^\lambda \ \approx \ \tau^\mu t_\mu^{\ \lambda}\sqrt{|g|} \ + \ d_\nu U^{\nu\lambda},$$

which is the desired decomposition. ■

This equation exhibits that the energy-momentum current $\mathcal{I}_A^\lambda$ differs by a total divergence from the metric energy-momentum $\tau^\mu t_\mu^\lambda\sqrt{|\det g|}$. Because $U^{\nu\lambda}$ is a superpotential (i.e. its divergence is a Noether current that itself reduces to a boundary term), the difference carries no independent local dynamics and does not affect the covariantly conserved quantity (3.43).

The passage from $\mathcal{I}_A^\lambda$ to t_μ^λ is the gauge-theoretic counterpart of the Belinfante-Rosenfeld construction in classical field theory, where one adds $\partial_\nu U^{\nu\lambda}{}_\mu$ (with $U^{\nu\lambda}{}_\mu$ antisymmetric in ν, λ) to the canonical tensor to obtain the symmetric metric tensor. The physically relevant energy-momentum tensor in gauge theory is t_μ^λ, while the Noether current $\mathcal{I}_A^\lambda$ can be viewed as its precursor, differing only by a total divergence.

In the Yang-Mills setting, the gauge-theoretic superpotential (3.31) plays the role of that antisymmetric improvement term. It removes the connection-dependent part of the current and yields the physically relevant, covariantly conserved metric energy-momentum tensor. Similarly, the current $\mathcal{I}_A^\lambda$ in (3.37) is analogous to the canonical stress-energy tensor (as it depends on the choice of a principal connection A used to define the lift $\widetilde{\tau}_A$) and the metric energy-momentum tensor t_μ^λ in (3.39)–(3.43) is analogous to the metric stress-energy tensor (being obtained from the metric variation of the Yang–Mills Lagrangian). The weak identity (3.39) shows that the difference between these two objects is exactly accounted for by a Noether current of the form (3.44), which, as we have seen, reduces to a superpotential.

References

[AJ78] M.Atiyah, J.D.S. Jones. *Topological Aspects of Yang-Mills Theory.* Comm. Math. Phys. 61, (1978), 97–118.

[CVB03] R. Cianci, S. Vignolo and D. Bruno. *The geometrical framework for Yang–Mills theories.* J. Phys. A: Math. Gen. 36 8341, 2003.

[DK97] S. Donaldson and P. Kronheimer. *The Geometry of Four-Manifolds.* Oxford Mathematical Monographs, Oxford University Press, 1997.

[H17] M. Hamilton. *Mathematical Gauge Theory, with Applications to the Standard Model of Particle Physics.* Springer, 2017.

[NS96] S. Nishikawa, R. Schoen. *Lectures on Geometric Variational Problems.* Springer, 1996.

[R15] T. Riviere, *The Variations of Yang–Mills Lagrangians*. 2015.

[T94] C.G. Torre. *Natural symmetries of the Yang-Mills equations*. J.Math.Phys. 36 (1995) 2113-2130.

[W94] E. Witten. *Supersymmetric Yang-Mills theory on a four-manifold.* J. Math. Phys. 35:5101-5135, 1994.