# Symmetries of Topological Graphs

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## Introduction

Graphs are commonly described as sets of points (vertices) coupled with a set of connections between the vertices (edges). Graphs can also be described as a set of vertices together with a disjoint union of open intervals embedded in $\mathbb{R}^3$, such that each interval is bounded by two distinct vertices. In this context the aforementioned intervals (edges) take on a physical meaning and can be studied as a topological space. These topological graphs will be our main focus of study. Our goals for this project are as follows:

- Learn about group theory and its common applications in topology and geometry.
- Characterize the homeomorphism group of a topological graph and describe its properties.
- Produce results parallel to Theorem 6.8 (Farb, Margalit) in the context of homology groups of topological graphs.

A topological graph is a pair consisting of a Hausdorff space $X$ and a finite subspace $V \subseteq X$ (called vertices) such that the following two conditions hold:

(i) $X - V$ is a finite disjoint union of open subsets $e_1, e_2 \ldots, e_n$ called edges, and each edge is homeomorphic to an open interval of the real line.

(ii) The boundary $\overline{e_i} - e_i$ of the edge $e_i$ consists of two distinct vertices, and the pair $(\overline{e_i}, e_i)$ is homeomorphic to the pair $([0, 1], (0, 1))$.

In particular, our graphs are finite and regular (without any loops connecting vertices to themselves). From now on we will use the notation $e_i$ to denote both the open edge and the closed edge depending on the context – this should not cause any confusion because continuous maps defined on an edge $e_i$ can be uniquely extended to a continuous map on the closure $\overline{e_i}$. Henceforth let $X$ denote a topological graph. We begin by stating some relevant definitions:

- The group of adjacency-preserving homeomorphisms of $X$ is denoted $\text{Homeo}(X)$, and for the sake of brevity we will simply call these homeomorphisms. In other words, this is really the group of cellular homeomorphisms of a one-dimensional cell complex.
- The group of orientation-preserving homeomorphisms of $X$ is denoted $\text{Homeo}^+(X)$.
- The group of graph automorphisms of $X$ is denoted $\text{Aut}(X)$.
- The group of edge automorphisms of $X$ is denoted $\text{Aut}^*(X)$.
- The boundary map $\partial : C_1(X) \to C_0(X)$ from the group of 1-chains to the group of 0-chains (free abelian groups generated by the edges and vertices of $X$, respectively) is defined by the formula $\partial(\text{edge}) = \text{final vertex} - \text{initial vertex}$.
- The first homology group of $X$ is defined as the group $H_1(X) = \ker \partial$. Intuitively, it measures the number of holes or cycles in $X$.
- Given a homeomorphism $f : X \to X$, the induced map on homology $f^* : H_1(X) \to H_1(X)$ is defined by the formula $f^*(\text{cycle}) = f(\text{cycle})$ (note that $f$ necessarily maps cycles to cycles).

# Results on the homeomorphism group

Since a topological graph $X$ (with say, $n$ edges) is basically a disjoint union of open intervals joined by vertices, any homeomorphism of $X$ can be thought of as a collection of homeomorphisms of the unit interval $I = [0, 1]$ glued together, and we envision the homeomorphism as acting on $X$ according to these homeomorphisms of intervals. How can we make this idea precise?

We follow the procedure described by Hatcher [4]. Since there is only one way to attach edges to each other via vertices once the adjacency relations of the graph are specified, we can associate to each edge a canonical choice of (orientation-preserving) homeomorphism $\varphi_i : e_i \to I$. The homeomorphisms $\varphi_1, \varphi_2, \ldots, \varphi_n$ are called ***characteristic maps*** for $X$. Now, given any $f \in \text{Homeo}(X)$, we can restrict to any edge $e_i$ to obtain a homeomorphism $f_i = f|_{e_i} : e_i \to e_{\sigma(i)}$ where $\sigma \in S_n$, and then factor through the appropriate characteristic maps to express $f_i$ as a homeomorphism of intervals.

Thus we obtain a collection of homeomorphisms $\widetilde{f}_i = \varphi_{\sigma(i)} \circ f_i \circ \varphi_i^{-1} \in \text{Homeo}(I)$, which we call ***interval representatives*** for $f$. Given two elements $f, g \in \text{Homeo}(X)$ which permute edges according to $\sigma, \tau \in S^n$ respectively, we define an operation

$$(\widetilde{f}_1, \widetilde{f}_2, \ldots, \widetilde{f}_n) \cdot (\widetilde{g}_1, \widetilde{g}_2, \ldots, \widetilde{g}_n) = (\widetilde{f}_{\tau(1)} \circ \widetilde{g}_1, \widetilde{f}_{\tau(2)} \circ \widetilde{g}_2, \ldots, \widetilde{f}_{\tau(n)} \circ \widetilde{g}_n).$$

It follows from calculations made in the proof of Theorem 1 below that $\text{Homeo}(I)^n = \text{Homeo}(I) \times \cdots \times \text{Homeo}(I)$ is a group with respect to this operation. We will summarize the preceding remarks in the following lemma:

**Lemma 1.** *Label the edges of $X$ by $\{e_1, e_2, \dots, e_n\}$ and let $f \in \text{Homeo}(X)$ permute edges according to $\sigma \in S_n$. Suppose that for each $1 \le i \le n$,*

*(i)* $\varphi_i : e_i \to I$ *denotes the $i$th characteristic map.*

*(ii)* $f_i = f|_{e_i} : e_i \to e_{\sigma(i)}$ *denotes the restriction of $f$ to the $i$th edge.*

*(iii)* $\widetilde{f}_i = \varphi_{\sigma(i)} \circ f_i \circ \varphi_i^{-1}$ *denotes the $i$th interval representative for $f$.*

*Then we have a group homomorphism $\psi : \text{Homeo}^+(X) \to (\text{Homeo}^+(I))^n$ given by*

$$\psi(f) = (\widetilde{f}_1, \widetilde{f}_2, \dots \widetilde{f}_n)$$

*Proof.* We first check that $\psi$ is a group homomorphism. Let $f, g \in \text{Homeo}^+(X)$ permute edges according to some permutations $\sigma, \tau \in S_n$ respectively. Then $f \circ g$ permutes edges according to $\sigma \circ \tau$, and the restriction of $f \circ g$ to edge $e_i$ is

$$\begin{aligned}(f \circ g)|_{e_i} &= f|_{e_{\tau(i)}} \circ g|_{e_i} \\ &= (\varphi^{-1}_{\sigma(\tau(i))} \circ \widetilde{f}_{\tau(i)} \circ \varphi_{\tau(i)}) \circ (\varphi^{-1}_{\tau(i)} \circ \widetilde{g}_i \circ \varphi_i) \\ &= (\varphi^{-1}_{\sigma(\tau(i))} \circ \widetilde{f}_{\tau(i)} \circ \widetilde{g}_i \circ \varphi_i)\end{aligned}$$

which means that the $i$th component of $\psi(f \circ g)$ is

$$\varphi_{\sigma(\tau(i))} \circ (f \circ g)|_{e_i} \circ \varphi_i^{-1} = \widetilde{f}_{\tau(i)} \circ \widetilde{g}_i$$

which is exactly the $i$th component of $\psi(f) \cdot \psi(g)$ by definition of the group operation in $\text{Homeo}^+(I)^n$. Hence $\psi(f \circ g) = \psi(f) \cdot \psi(g)$.

Now we verify that $\psi$ is surjective. This basically amounts to the statement that any sequence of orientation-preserving homeomorphisms of $I$ can be considered as the interval representatives for some orientation-preserving homeomorphism of $X$. Let $(g_1, g_2, \dots, g_n) \in \text{Homeo}^+(I)^n$, and define let $f \in \text{Homeo}^+(X)$ by setting

$$f_i = f|_{e_i} = \varphi_i^{-1} \circ g_i \circ \varphi_i$$

i.e. the map that sends each each edge $e_i$ to itself via the interval representative $g_i$. Each $f_i$ is an orientation-preserving homeomorphism as a composition of such maps, so the resulting map $f$ (obtained by applying the gluing lemma to the $f_i$ which agree on their common domains, for example) is an orientation-preserving homeomorphism of $X$. Evidently we have by definition,

$$\begin{aligned}\psi(f) &= (\varphi_1 \circ f_1 \circ \varphi_1^{-1}, \varphi_2 \circ f_2 \circ \varphi_2^{-1}, \dots, \varphi_n \circ f_n \circ \varphi_n^{-1}) \\ &= (g_1, g_2, \dots, g_n)\end{aligned}$$

so $\psi$ is surjective. ∎

Notice that the map $\psi$ in Lemma 1 is not injective because in the proof we chose to send each $e_i$ to itself, and this choice was arbitrary. In general it seems as though we can send each $e_i$ to $e_{\sigma(i)}$ for some $\sigma \in S_n$ as long as $\sigma$ defines a valid permutation of the edges of $X$, so that any two elements in a fiber differ by an element of $\mathrm{Aut}^*(X)$. This intuition will be confirmed by Theorem 1.

Furthermore, we note here that it was important to restrict our attention in Lemma 1 to orientation-preserving homeomorphisms, because the analogous map $\mathrm{Homeo}(X) \to \mathrm{Homeo}(I)^n$ without this restriction isn't surjective. This can be seen by considering the graph with just two edges joined by a vertex – in this situation there is no homeomorphism of the graph whose interval representation is the pair $(1-x, 1-x)$, because trying to flip both edges would result in a map that fails to be continuous.

**Lemma 2.** *There exists a surjective group homomorphism* $\phi : \mathrm{Homeo}^+(X) \to \mathrm{Aut}^*(X)$ *with* $\ker \phi \simeq \mathrm{Homeo}^+(I)^n$ *where* $n$ *is the number of edges in* $X$*. In particular, there is an isomorphism*

$$\mathrm{Homeo}^+(X)/\,\mathrm{Homeo}^+(I)^n \simeq \mathrm{Aut}^*(X).$$

*Proof.* By our assumption that any (orientation-preserving) homeomorphism maps edges homeomorphically onto edges, for each edge $e_i$ we have $f(e_i) = e_j = \alpha(e_i)$ for some edge automorphism $\alpha \in \mathrm{Aut}^*(X)$. Thus we define the function $\phi$ by $\phi(f) = \alpha$. This is clearly a group homomorphism because if $g \in \mathrm{Homeo}^+(X)$ permutes edges according to $\beta \in \mathrm{Aut}^*(X)$, and then $f \in \mathrm{Homeo}^+(X)$ permutes edges according to $\alpha \in \mathrm{Aut}^*(X)$, then the resulting composition $f \circ g$ permutes edges according to $\alpha \circ \beta$; i.e. $\phi(f \circ g) = \phi(f) \circ \phi(g)$.

Now we prove that $\phi$ is surjective. Fix an arbitrary $\alpha \in \mathrm{Aut}^*(X)$ and write $\alpha(e_i) = e_{\alpha(i)}$ for brevity. For any edge $e_i$ let $\varphi_i : e_i \to I$ denote the corresponding characteristic map. It is easy to show that $\varphi_i$ must map the endpoint vertices of $e_i$ to the endpoints of $I$, i.e. 0 and 1, and these values are immediately determined by our assumption that $\varphi_i$ is orientation-preserving. We define a function $f : X \to X$ by setting $f_i = f|_{e_i} = \varphi_{\alpha(i)}^{-1} \circ \varphi_i$ for each edge $e_i$; that is, $f$ permutes edges according to $\alpha$ and each of its interval representatives is id. We will check that this defines a homeomorphism of $X$ such that $\varphi(f) = \alpha$. The map $f$ is an orientation-preserving homeomorphism because it is the result of gluing together the orientation-preserving homeomorphisms $f_i$ along their common domains (on which they all agree). Moreover, we have for each edge $e_i$,

$$f(e_i) = \varphi_{\alpha(i)}^{-1}(\varphi_i(e_i)) = \varphi_{\alpha(i)}^{-1}(I) = \alpha(e_i)$$

so $\phi(f) = \alpha$ by definition of $\phi$.

Finally, we observe that $\ker \phi$ consists of all (orientation-preserving) homeomorphisms which map edges of $X$ to themselves. This subgroup can be identified with $\mathrm{Homeo}^+(I)^n$ because any such homeomorphism is uniquely determined by $n$ (orientation-preserving) homeomorphisms of the interval, i.e. the interval representatives of each $f_i$. The last conclusion then follows from the first isomorphism theorem. ■

Combining lemmas 1 and 2, we obtain a short exact sequence

$$1 \longrightarrow \mathrm{Homeo}^+(I)^n \xrightarrow{i} \mathrm{Homeo}^+(X) \xrightarrow{\phi} \mathrm{Aut}^*(X) \to 1$$

where $i$ denotes the inclusion (identifying $\mathrm{Homeo}^+(I)^n$ with the kernel of $\phi$). Moreover, this sequence is left-split because $\psi \circ i$ yields the identity automorphism of $\mathrm{Homeo}^+(I)^n$, and we immediately obtain an isomorphism of $\mathrm{Homeo}^+(X)$ with the direct product of the outer groups,

$$\mathrm{Homeo}^+(X) \simeq \mathrm{Homeo}^+(I)^n \times \mathrm{Aut}^*(X).$$

The only thing we need to prove is that $\mathrm{Homeo}^+(I)^n$ is a group with respect to the operation we defined at the beginning of this section. We summarize this result in the following theorem.

**Theorem 1.** *Suppose $X$ has $n$ total edges, indexed from 1 to $n$. Then $\mathrm{Homeo}^+(X)$ is isomorphic to the group described as follows:*

- *The underlying set is the Cartesian product $(\mathrm{Homeo}^+(I))^n \times \mathrm{Aut}^*(X)$.*
- *Given two elements $F = (\widetilde{f}_1, \widetilde{f}_2, \ldots, \widetilde{f}_n, \sigma)$ and $G = (\widetilde{g}_1, \widetilde{g}_2, \ldots, \widetilde{g}_n, \tau)$, the group product is defined as $F \cdot G = (\widetilde{f}_{\tau(1)} \circ \widetilde{g}_1, \widetilde{f}_{\tau(2)} \circ \widetilde{g}_2, \ldots, \widetilde{f}_{\tau(n)} \circ \widetilde{g}_n, \sigma \circ \tau)$.*

The isomorphism in the statement of the theorem stems from the idea of creating an element of $\mathrm{Homeo}^+(X)$ by applying an automorphism $\sigma$ to the graph's edges and each $\widetilde{f}_i$ to the edge indexed $i$.

*Proof.* The isomorphism follows immediately from the aforementioned left-split short exact sequence, so we just need to check that $\mathrm{Homeo}^+(I)^n \times \mathrm{Aut}^*(X)$ is a group with respect to the stated operation. It is clear that the element which is identity in each component behaves as the identity element with respect to this operation, and that inverses can be produced by replacing each component of an element by its respective inverse. Thus it will suffice to show that the operation is associative. Take three arbitrary elements

$$\begin{aligned} F &= (\widetilde{f}_1, \widetilde{f}_2, \ldots, \widetilde{f}_n, \sigma) \\ G &= (\widetilde{g}_1, \widetilde{g}_2, \ldots, \widetilde{g}_n, \tau) \\ H &= (\widetilde{h}_1, \widetilde{h}_2, \ldots, \widetilde{h}_n, \theta) \end{aligned}$$

and notice that the product $(F \cdot G) \cdot H$ certainly agrees with the product $F \cdot (G \cdot H)$ in its last component, since this is just the product in $\mathrm{Aut}^*(X)$ which is associative. Thus we will focus on the first $n$ components. For $1 \leq i \leq n$ the $i$th component of $F \cdot G$ is

$$(F \cdot G)_i = \widetilde{f}_{\tau(i)} \circ \widetilde{g}_i$$

and therefore the $i$th component of $(F \cdot G) \cdot H$ is

$$((F \cdot G) \cdot H)_i = (\widetilde{f}_{\tau(\theta(i))} \circ g_{\theta(i)}) \circ \widetilde{h}_i$$

Similarly, the $i$th component of $G \cdot H$ is

$$(G \cdot H)_i = \widetilde{g}_{\theta(i)} \circ \widetilde{h}_i$$

and therefore the $i$th component of $F \cdot (G \cdot H)$ is

$$(F \cdot (G \cdot H))_i = \widetilde{f}_{\tau(\theta(i))} \circ (\widetilde{g}_{\theta(i)} \circ \widetilde{h}_i)$$

and since composition is associative in each Homeo$(I)$ component, a direct comparison shows that $(F \cdot G) \cdot H = F \cdot (G \cdot H)$ as desired. ■

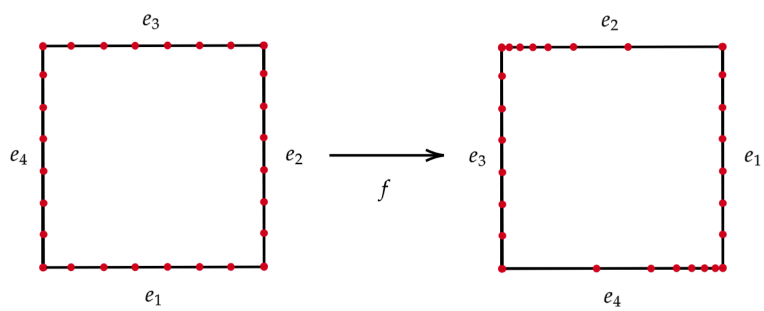


Figure 1: An example of a homeomorphism from the cycle graph $C_4$ to itself.

**Example.** The homeomorphism $f$ in Figure 1 is a 90-degree counterclockwise rotation mapping $e_4$ and $e_2$ according to the interval representative $x^2$ and $e_1$ and $e_3$ according to the identity. Hence we identify $f$ with the tuple $(x, x^2, x, x^2, \sigma)$ where $\sigma = (1234)$.

A well-known fact about homeomorphisms $I \to I$ is that they are either monotonically increasing (in which case they fix the endpoints) or decreasing (in which case they swap the endpoints). From this it follows that there are no nontrivial finite-order orientation-preserving homeomorphisms of $I$, and that any finite-order orientation-reversing homeomorphism of $I$ is conjugate to the linear involution $x \mapsto 1-x$. Combining this with the result of Theorem 1, we have the following classification of finite-order elements of Homeo$(X)$.

**Corollary 1** (Classification of finite order elements)**.** *If $f$ is a finite-order element of* Homeo$(X)$*, then the order of $f$ equals the order of its corresponding automorphism. Moreover, any finite-order element of* Homeo$(X)$ *can be generated as follows:*

1. *Fix an automorphism $\sigma$ of the vertices of $X$.*

2. *Use this to directly generate an edge automorphism, call it $\sigma$ again.*

3. *Decompose $\sigma$ into its permutation cycles, i.e. the cyclical sequence of edges obtained by repeatedly applying $\sigma$ to an edge (not to be confused with graph cycles).*

4. *For each permutation cycle, pick arbitrary elements of* $\mathrm{Homeo}^+(I)$ *for all but one edge, the element for the final edge is determined as the inverse of the composition of all previously chosen elements in that cycle.*

5. *Repeat the above step for all permutation cycles. The resulting list of elements of* $\mathrm{Homeo}^+(I)$ *together with the automorphism* $\sigma$ *fully determine a finite order graph homeomorphism by the previous theorem.*

We previously remarked (in the paragraph following Lemma 1) that the homomorphism $\psi$ cannot be extended to a homomorphism $\mathrm{Homeo}(X) \to \mathrm{Homeo}(I)^n$. Then the question arises: what can we say about the larger group $\mathrm{Homeo}(X)$? In fact, the analogue of Lemma 2 still holds in this slightly more general situation.

**Lemma 3.** *There exists a surjective group homomorphism* $\phi : \mathrm{Homeo}(X) \to \mathrm{Aut}(X)$ *with* $\ker \phi \simeq \mathrm{Homeo}^+(I)^n$*, where* $n$ *is the number of edges in* $X$*. In particular, there is an isomorphism*

$$\mathrm{Homeo}(X)/\,\mathrm{Homeo}^+(I)^n \simeq \mathrm{Aut}(X)$$

*Proof.* For any $f \in \mathrm{Homeo}(X)$, define $\phi(f) = f|_V$ where $V$ denotes the set of vertices of $X$. Note that for any vertex $v_0$ we must have $\phi(f(v_0)) = f(v_0)$, because $v_0$ is in the domain of both $\phi(f)$ and $f$. Therefore $(\phi(f) \circ \phi(g))(v_0) = (\phi(f))(g(v_0)) = \phi(f \circ g)(v_0)$. Hence $\phi$ is a group homomorphism.

Now we will check that $\phi$ is surjective. We fix an ordering of the vertices of $X$, say $e_i = (a_{i,0}, a_{i,1})$ for each $1 \leq i \leq n$. Let $\sigma \in \mathrm{Aut}(X)$ be any graph automorphism. The automorphism $\sigma$ determines a unique edge automorphism which we also label $\sigma$, so that $\sigma$ defines a permutation of the edges $\sigma(e_i) = e_{\sigma(i)}$. We define a homeomorphism $f : X \to X$ as follows:

1. If $v$ is an isolated vertex (with degree 0) then set $f(v) = \sigma(v)$.

2. If $\sigma(a_{i,0}) = a_{\sigma(i),0}$ then set $\widetilde{f}_i = \mathrm{id} : I \to I$.

3. If $\sigma(a_{i,0}) = a_{\sigma(i),1}$ then set $\widetilde{f}_i = j : I \to I$.

4. Set $f|_{e_i} = f_i = \varphi^{-1}_{\sigma(i)} \circ \widetilde{f}_i \circ \varphi_i$.

where $j(x) = 1 - x$ is the linear involution of $I$. We claim that the function $f$ obtained by gluing together the maps $f_i$ is a homeomorphism satisfying $f|_V = \sigma$. It is clear that any isolated vertices must map to other isolated vertices under both $f$ and $\sigma$, and that $f$ and $\sigma$ agree on these vertices by definition. Thus it will suffice to check that $f$ and $\sigma$ agree at the endpoints of every edge.

We need to check four cases, according to whether the given vertex is the 0-vertex or the 1-vertex of an edge, and whether $\sigma$ follows the second or third condition for the given vertex. Of course, these are all essentially the same quick calculation. In each case we use the fact that $\varphi_i(a_{i,0}) = 0$ and $\varphi_i(a_{i,1}) = 1$ by our orientation-preserving convention.

**Case (i).** If $\sigma(a_{i,0}) = a_{\sigma(i),0}$ then

$$f(a_{i,0}) = f_i(a_{i,0}) = \varphi_{\sigma(i)}^{-1} \circ \varphi_i(a_{i,0}) = \varphi_{\sigma(i)}^{-1}(0) = a_{\sigma(i),0} = \sigma(a_{i,0}).$$

**Case (ii).** If $\sigma(a_{i,1}) = a_{\sigma(i),1}$ then $\sigma(a_{i,0}) = a_{\sigma(i),0}$ as well, so

$$f(a_{i,1}) = f_i(a_{i,1}) = \varphi_{\sigma(i)}^{-1} \circ \varphi_i(a_{i,1}) = \varphi_{\sigma(i)}^{-1}(1) = a_{\sigma(i),1} = \sigma(a_{i,1}).$$

**Case (iii).** If $\sigma(a_{i,0}) = a_{\sigma(i),1}$ then $\sigma(a_{i,1}) = a_{\sigma(i),0}$ as well, so

$$f(a_{i,0}) = f_i(a_{i,0}) = \varphi_{\sigma(i)}^{-1} \circ j \circ \varphi_i(a_{i,0}) = \varphi_{\sigma(i)}^{-1}(j(0)) = \varphi_{\sigma(i)}^{-1}(1) = a_{\sigma(i),1} = \sigma(a_{i,0}).$$

**Case (iv).** If $\sigma(a_{i,1}) = a_{\sigma(i),0}$ then

$$f(a_{i,1}) = f_i(a_{i,1}) = \varphi_{\sigma(i)}^{-1} \circ j \circ \varphi_i(a_{i,1}) = \varphi_{\sigma(i)}^{-1}(j(1)) = \varphi_{\sigma(i)}^{-1}(0) = a_{\sigma(i),0} = \sigma(a_{i,1}).$$

Finally, $\ker \phi$ is characterized in exactly the same way as in Lemma 2. It consists of all homeomorphisms which fix the vertices of $X$, and since a homeomorphism which fixes every vertex must also fix every edge, the subgroup $\ker \phi$ can be identified with $\mathrm{Homeo}^+(I)^n$ in the same way as before. ■

As a result of Lemma 3 we have a short exact sequence

$$1 \longrightarrow \mathrm{Homeo}^+(I)^n \xrightarrow{i} \mathrm{Homeo}(X) \xrightarrow{\phi} \mathrm{Aut}(X) \to 1$$

where $i$ again denotes the inclusion after identifying $\mathrm{Homeo}^+(I)^n$ with the kernel of $\phi$. Due to the equivalence between short exact sequences and semidirect products, we obtain the following characterization of the group $\mathrm{Homeo}(X)$.

**Theorem 2.** *Suppose $X$ has $n$ total edges. Then* $\mathrm{Homeo}(X)$ *is a semidirect product*

$$\mathrm{Homeo}(X) \simeq \mathrm{Homeo}^+(I)^n \rtimes_\xi \mathrm{Aut}(X)$$

*with respect to some homomorphism $\xi : \mathrm{Aut}(X) \to \mathrm{Aut}(\mathrm{Homeo}(I)^n)$.*

The story is different from before, however, because this short exact sequence is in general *not* left-split. We cannot use $\psi$ to split the sequence this time because, again, it does not map $\mathrm{Homeo}(X)$ onto $\mathrm{Homeo}^+(I)^n$. Consider the most simple graph $X = I$, and suppose that we have a group homomorphism $p : \mathrm{Homeo}(I) \to \mathrm{Homeo}^+(I)$ such that $p \circ i = \mathrm{id} : \mathrm{Homeo}^+(I) \to \mathrm{Homeo}^+(I)$. This latter condition means that for any orientation-preserving homeomorphism $f : I \to I$, we have $p(f) = f$. For any orientation-reversing homeomorphism $g : I \to I$, $g \circ j$ is orientation-preserving so $g \circ j = p(g \circ j) = p(g) \circ p(j)$. Moreover, we have $x = p(x) = p(j \circ j) = p(j) \circ p(j)$ and since $p(j)$ is orientation-preserving the only possibility is $p(j) = x$. Therefore $p$ is completely determined by the equation $p(g) = g \circ j$.

On the other hand, if $g$ has finite order then there exists an orientation-preserving homeomorphism $f$ such that $g = f^{-1} \circ j \circ f$. Hence in this case

$$p(g) = p(f^{-1}) \circ p(j) \circ p(f) = f^{-1} \circ x \circ f = x$$

The two equations $p(g) = g \circ j$ – which holds in general – and $p(g) = x$ – which holds for finite order homeomorphisms – cannot simultaneously be true. For example, if we take $f(x) = x^2$ and $g(x) = f^{-1} \circ j \circ f(x) = \sqrt{1-x^2}$, then since $g$ has finite order we have

$$p(g) = g \circ j = \sqrt{1-(1-x)^2} \neq x.$$

This contradiction implies that our supposition was erroneous, and there is no homomorphism $p$ which splits the short exact sequence. Frankly speaking, the authors do not think that Theorem 3 is very practically useful; however, we do find this to be an interesting geometric example of a short exact sequence which is right-split but not left-split. Even more interesting, we think, is that the difference between the group of orientation-preserving homeomorphisms, and the group of homeomorphisms in general, is encoded precisely by the splitting and respectively failed splitting of their short exact sequences.

## Homeomorphisms acting on the homology group

Our motivation is the following well-known result about mapping class groups of surfaces.

**Theorem 3** (Farb, Margalit Theorem 6.8 [1])**.** *For a surface $S$ of genus $g$, let $\mathrm{Mod}(S)$ denote the mapping class group of $S$ (consisting of isotopy classes of orientation-preserving homeomorphisms of $S$). Suppose $f \in \mathrm{Mod}(S)$ has finite order and $f^* \in \mathrm{Aut}(H_1(S))$ is induced from $f$. Then $f^* = \mathrm{id}$ implies that $f = \mathrm{id}$.*

**Topological graph version of Theorem 3.** *Let $X$ be a connected topological graph with the property that any nontrivial finite homeomorphism $f \in \mathrm{Homeo}(X)$ induces a nontrivial homomorphism $f^* \in \mathrm{Aut}(H_1(X))$. Then we will say that "the graph $X$ satisfies Theorem 3," or "Theorem 3 holds for $X$".*

Notice that Theorem 3 does not hold universally if we simply replace the surface $S$ by a (connected) topological graph $X$. For example, a rotation of a cycle or a map that permutes bridges and leaves the cycles unchanged would be a nontrivial homeomorphism inducing the identity map on homology. We would like to provide a partial characterization of topological graphs where the equivalent of Theorem 3 holds true. The following key lemma simplifies this to a problem of graph theory, where some results are already known (Sunada Theorem 4.5 [7], for example).

**Lemma 4.** *$F \in \mathrm{Homeo}(X)$ is a counterexample to the topological graph version of Theorem 3 if and only if $F|_V$ is a counterexample to the (non-topological) graph version of Theorem 3.*

*In other words: there is a nontrivial finite order homeomorphism $F$ inducing $F^* = \mathrm{id}$ if and only if the automorphism corresponding to $F$ is also nontrivial and induces* id.

*In other words again: the problem of characterizing nontrivial induced automorphisms on homology is the same for graphs and topological graphs.*

*Proof.* Throughout this proof, to say that a function is a counterexample to the graph-theoretic analogue of Theorem 3 (in either the topological or non-topological sense), means that the function is nontrivial, it has finite order, and that it induces the identity map on homology. We will also use the classification of finite order elements stated in Corollary 1.

We start with the "if" direction. We claim that if $f$ is a graph automorphism that serves as a counterexample to Theorem 3 in the non-topological sense, then the homeomorphism $F = (\mathrm{id}, \mathrm{id}, \ldots, \mathrm{id}, f)$ is a counterexample in the topological sense. Note that if edge $e_i$ maps to edge $e_j$ with some orientation under $f$, then it must do so in the same way under $F$, so it is clear that $F^*$ is the identity map whenever $f^*$ is the identity. The same logic shows that if $f$ is not itself the identity map, then clearly $F$ is also not the identity map. Further, one can verify that $F^k = (\mathrm{id}, \mathrm{id}, \ldots, \mathrm{id}, f^k)$, and since $f$ has finite order $F$ is also of finite order. Therefore, if $f$ is a counterexample, then so is $F$.

For the "only if" direction, suppose that $F$ is a finite-order homeomorphism serving as a counterexample to Theorem 3 for topological graphs. We claim that $f$, the automorphism corresponding to $F$, is a counterexample in the non-topological sense. First note that $f$ definitely has finite order since it's an element of a finite group (of automorphisms of a finite graph). By the same reasoning as before, if $F^*$ is the identity, then so is $f^*$, because the only component of $F$ determining which edges map to which other edges is its corresponding automorphism, which is $f$ by definition.

To prove that $f$ is nontrivial whenever $F$ is nontrivial, we take the contrapositive. If $f = \mathrm{id}$, then each edge maps to itself, so the homeomorphism $F$ is uniquely determined by its interval representatives $\widetilde{f}_1, \widetilde{f}_2, \ldots$ and moreover in this case we have $F^k = (\widetilde{f}_1^k, \widetilde{f}_2^k, \ldots)$. Therefore, we see that if $F$ has finite order, then each $\widetilde{f}_i$ must also be of finite order, but the only orientation-preserving finite order homeomorphism of $I$ is the identity map. Hence $\widetilde{f}_i = \mathrm{id}$ for each $i$, implying that $F = \mathrm{id}$, and the contrapositive is proved.

We have shown that there is a counterexample to Theorem 3 in the topological case if and only if there is a counterexample in the non-topological case, proving that the two problems are equivalent. ■

We can combine the previous lemma with Sunada Theorem 4.5 [7], which states that for a connected graph which is not a cycle and has no bridges, any nontrivial automorphism induces a nontrivial map on homology. We immediately find that Theorem 3 holds for a connected topological graph which is not a cycle and has no bridges. The following theorem also provides a significant expansion of the result proven by Sunada.

**Theorem 4.** *If a connected topological graph is not a cycle and has no bridges, then any nontrivial homeomorphism of the graph induces a nontrivial automorphism of its homology group.*

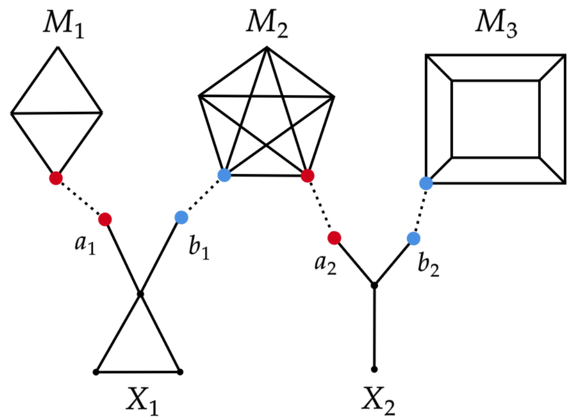


Figure 2: Illustration of condition (iv) in Theorem 4. The vertices $a_i$ and $b_i$ must be fixed when joined to $M$, which pins down the rest of $X_i$.

*There are several other conditions ensuring that nontrivial homeomorphisms induce nontrivial automorphisms:*

*(i) Suppose graph $G$ satisfies Theorem 3 and consists of a disjoint union of connected subgraphs $\{G_1, G_2 \cdots\}$. Then each graph $G_i$ satisfies Theorem 3.*

*(ii) Conversely, suppose $H = \{H_1, H_2 \cdots\}$ is a set of disjoint connected graphs with each $H_i$ satisfying Theorem 3, and no two trees in $H$ are isomorphic. Then the collective graph $H$ satisfies Theorem 3. Note the additional clause in the converse that was not present in condition (i).*

*(iii) Suppose a graph $M$ is not a cycle and has no bridges, and suppose each $X_i$ in a set of graphs $\{X_1, X_2 \cdots\}$ contains a vertex $a_i$ with the property that the only automorphism of $X_i$ that fixes $a_i$ is* id *(henceforth, call a graph with this property* ***appendable****). Then the graph produced by appending each $X_i$ to a unique vertex of $M$ using $a_i$ satisfies Theorem 3.*

*(iv) Suppose a set of graphs $M = \{M_1, M_2 \cdots\}$ have no bridges (however, they could be cycles). Further, let $X = \{X_1, X_2 \cdots\}$ be a set of multiple connected graphs with the property that: Each $X_i$ contains two vertices $a_i$ and $b_i$ such that the only automorphism of $X_i$ fixing both $a_i$ and $b_i$ and inducing the trivial map on homology is* id*. Then any connected graph formed by connecting elements of $M$ to each other via elements of $X$ using the endpoints $a, b$ satisfies Theorem 3 (see Figure 2).*

*(v) Suppose $C$ is a cycle and $\{X_1, X_2 \cdots\}$ is a set of appendable trees. Then appending these trees to unique vertices on $C$ results in a graph that satisfies Theorem 3 if and only if it does not have any nontrivial rotational symmetry about $C$ (see Figure 3).*

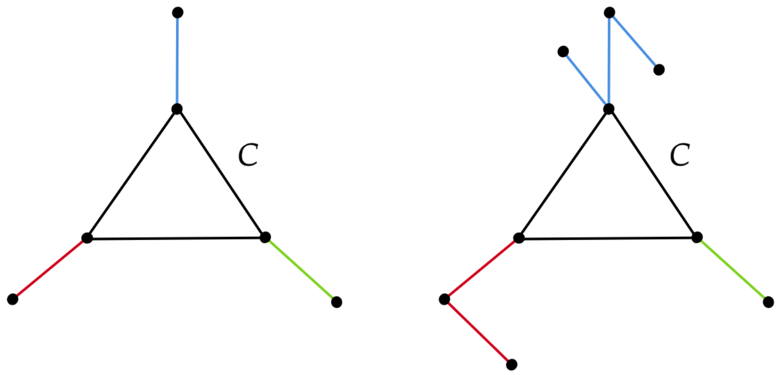


Figure 3: Illustration of condition (v) in Theorem 4. The cycle $C$ on the left has trees appended in a rotationally symmetric way, whereas the graph on the right does not.

In proving conditions (i)-(v), we will use the following lemma which is a direct consequence of the first part of Theorem 4.

**Lemma 5.** *If a subgraph $Y$ of a topological graph $X$ has no bridges and is not a cycle, then for any $f \in \mathrm{Homeo}(X)$ we have $f^* = \mathrm{id}$ implies that $f|_Y = \mathrm{id}$.*

**Proof of Condition (i) in Theorem 4.** We take the contrapositive. Suppose one of the subgraphs $G_i$ did not satisfy Theorem 3. Then there exists an automorphism $f_i$ of $G_i$ that is a counterexample to Theorem 3. Define an automorphism $f$ of $G$ by $f(x) = f_i(x)$ if $x$ is in $G_i$ and $f(x) = \mathrm{id}$ otherwise. Then clearly $f$ is a counterexample on $G$.

**Proof of Condition (ii) in Theorem 4.** Note that each connected component $H_i$ must map completely onto another connected component $H_j$ under any automorphism of $H$, so the image of any subset of $H_i$ fully determines the image of $H_i$. If an automorphism $f$ induces the identity map on the homology group of $H$, then clearly any $H_i$ containing some cycle $C_i$ must map to itself because $C_i$ must map to itself. Thus, $f$ restricted to $H_i$ is an automorphism of $H_i$, meaning it must be the identity (because $H_i$ satisfies Theorem 3 by assumption). Further, because no trees in $H$ are isomorphic, each tree must map to itself, and further must do so by the identity automorphism (because each tree in $H$ also satisfies Theorem 3 by assumption). Because each element of $H$ is either a tree or has a cycle, we are done.

**Proof of Condition (iii) in Theorem 4.** The idea of this condition is that appending some graph $X_i$ to $M$ forces the vertex $a_i$ to be fixed, which pins down the rest of $X_i$ by the appendability assumption. By Lemma 5, every vertex in $M$ must map to itself. Because each vertex $a_i$ coincides with a vertex of $M$ by construction, each $a_i$ must be fixed, and by the definition of appendability, each $X_i$ must also be fixed.

**Proof of Condition (iv) in Theorem 4.** The idea here is similar to that of condition (iii). Suppose $M_c$ and $M_d$ have no bridges and they are joined together by some connected graph $X$ satisfying the condition stated in (iv). Let $f$ be an automorphism with $f^* = \text{id}$. If either $M_c$ or $M_d$ is not a cycle then it must be fixed by $f$ by Lemma 5. Otherwise if either of $M_c$ or $M_d$ is a cycle, say $M_c$, then the only nontrivial possibility for $f$ is that it rotates $M_c$ since any reflection will transform its homology nontrivially. But then the vertex $a_i$ which joins $M_c$ to $X_i$ would leave $X_i$, contradicting the fact that $f$ preserves adjacency. Therefore $f$ must fix any cycles, and it must fix $X_i$ by the appendability assumption, so $f = \text{id}$ and we are done.

**Proof of Condition (v) in Theorem 4.** The idea of this condition is that any cycle with trees appended in a rotationally symmetric way will have nontrivial rotations which induce the trivial map on homology, but this possibility is ruled out when we introduce asymmetries.

If there is a rotational symmetry about $C$, then simply performing that rotation will provide a counterexample to Theorem 3. Otherwise, every rotation of elements of $C$ is ruled out due to the lack of rotational symmetry, and thus the only transformation of $C$ is fixing each vertex. Then, by the definition of appendability, each $X_i$ must also be fixed.

# Directions for future research

Here is a list of some of our unanswered questions that could inspire future research:

1. How can we use the cellular approximation theorem to transfer information about the group of cellular homeomorphisms into information about the group of homeomorphisms of a topological graph? In other words, how does the algebraic structure of the group of homeomorphisms interact with the topological approximation of a homeomorphism by cellular homeomorphisms?

2. Can we provide a stronger classification of graphs that satisfy Theorem 3? If not, can we prove that all such graphs fall into one of our given categories?

3. Grossman proved in 1975 that the group of homeomorphisms of a compact surface is residually finite (Theorem 6.11 in Farb and Margalit). Thus we might ask: is the group $\text{Homeo}(X)$ of homeomorphisms of a topological graph residually finite? An affirmative solution would lead to some nice results regarding the existence of finite quotients of $\text{Homeo}(X)$.

4. Usually for a topological graph, the higher homology groups $H_2(X), H_3(X), \ldots$ are all zero; however, one can construct a complex where these groups are nonzero (called the flag complex or clique complex) by thinking of higher-dimensional cells as complete subgraphs. Then we might ask: how can we classify the finite-order homeomorphisms of $X$ which act nontrivially on $H_k(X)$ for $k > 1$?

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