From differential operators to geometric algebras

Alex Taylor
The University of Illinois at Urbana-Champaign

Abstract

We explore the relationship between the existence of Dirac operators on a vector bundle E and Clifford module structures on E. We define Dirac operators on Euclidean space and then generalize this to Dirac-type operators acting on sections of a vector bundle. We construct the Clifford algebra, prove the universal property, and then generalize this to the Clifford bundle structure on a vector bundle. Then we prove that there exists Dirac-type operators on a vector bundle if and only if that vector bundle admits algebra-representations on each fiber, i.e. a Clifford module structure.

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1 Introduction

In special relativity, the energy E of a free particle with rest mass m and momentum p satisfies the equation

$$E^2 = (|p|c)^2 + (mc^2)^2$$

which is the so-called *energy-momentum relation*. Writing $p = (p_1, p_2, p_3)$ for a particle in $\mathbb{R}^3$ and solving for energy yields

$$E = c\sqrt{m^2c^2 + p_1^2 + p_2^2 + p_3^2}$$

For simplicity we can rescale our coordinates so that $c = \hbar = 1$, and in this case the equation becomes

$$E = \sqrt{m^2 + p_1^2 + p_2^2 + p_3^2}$$

When this equation is "quantized", E and p_j are replaced by their corresponding quantum mechanical operators, $i\frac{\partial}{\partial t}$ and $-i\frac{\partial}{\partial x_j}$ respectively. Hence we get

$$i\frac{\partial}{\partial t} = \sqrt{m^2 - \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_3^2}} = \sqrt{m^2 + \Delta}$$

where $\Delta = -\sum_{j=1}^{3} \frac{\partial^2}{\partial x_j^2}$ is the Laplacian on $\mathbb{R}^3$. So given any state function $\psi(x,t)$ for the particle, the quantum mechanics are described by

$$i\frac{\partial \psi}{\partial t} = \left(\sqrt{m^2 + \Delta}\right)\psi$$

and thus the question naturally arises: what is the square root of the Laplacian Δ? In other words, we are searching for a constant coefficient first-order differential operator D satisfying $D^2 = m^2 + \Delta$, so that

$$i\frac{\partial \psi}{\partial t} = D\psi(x,t)$$

To figure out what D should be, let's write it in components

$$D = m\gamma_0 + \sum_{j=1}^{3} \gamma_j \frac{\partial}{\partial x_j}$$

Then the relation $D^2 = m^2 + \Delta$ means that

$$\left(m\gamma_0 + \gamma_1\frac{\partial}{\partial x_1} + \gamma_2\frac{\partial}{\partial x_2} + \gamma_3\frac{\partial}{\partial x_3}\right)^2 = m^2 - \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_3^2}$$

Now expanding the left-hand side of this equation and comparing coefficients yields the system of equations:

$$\begin{cases} \gamma_0^2 = 1, \\ \gamma_1^2 = \gamma_2^2 = \gamma_3^2 = -1, \\ \gamma_j\gamma_k + \gamma_k\gamma_j = 0 \quad \text{for } 0 \le j \ne k \le 3 \end{cases} \tag{1}$$

Obviously there are no complex numbers $\gamma_0, \gamma_1, \gamma_2, \gamma_3$ satisfying this system; however, it is easy to find four 4×4 complex matrices that do solve it. Namely, the ***Dirac matrices***:

$$\gamma_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \qquad \gamma_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$$

$$\gamma_2 = \begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix} \qquad \gamma_3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

In any case, the important thing about these matrices is not their entries themselves but rather the algebraic relations (1) that they satisfy. Notice that we can write the system (1) in the succinct form

$$\gamma_j\gamma_k + \gamma_k\gamma_j = \begin{cases} 2I_4 & \text{if } j = k = 0 \\ -2I_4 & \text{if } 1 \leq j = k \leq 3 \\ 0 & \text{if } j \neq k \end{cases}$$

Or, what is the same,

$$\gamma_j\gamma_k + \gamma_k\gamma_j = (-2q_{jk})I_4 \tag{2}$$

where q is the bilinear form given by

$$q(u, v) = u^T \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} v = u^T \eta v$$

Equation (2) is precisely the defining relation for the Clifford algebra structure on the tensor algebra of a vector space equipped with a bilinear form q. In the following section we will explain exactly what this means, and we will take this as our motivation for studying the connection between Dirac operators, which satisfy $D^2 = \Delta$, and the algebraic structure (2) on the tangent spaces of a Riemannian manifold.

Remark. In equation (2) we are using the standard notation $q_{jk} = q(e_j, e_k)$ where e_j is a fixed basis for the vector space. In this case the bilinear form q is the usual metric on Minkowski spacetime $\mathbb{R} \times \mathbb{R}^3$.

2 Clifford algebras

Let V be a vector space over $\mathbb{C}$ equipped with a bilinear form $q : V \times V \to \mathbb{C}$. The ***tensor algebra*** generated by V is the vector space

$$\bigotimes V = \mathbb{C} \oplus V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \cdots$$

which is a unital associative algebra over $\mathbb{C}$ with respect to the tensor product operation on vectors. Elements of the tensor algebra look like

$$a_0 + a_1 v_1 + a_2(v_{2,1} \otimes v_{2,2}) + \cdots + a_n(v_{n,1} \otimes \cdots \otimes v_{n,n})$$

where $a_j \in \mathbb{C}$ for each $1 \leq j \leq n$ and $v_{j,k} \in V$ for each $1 \leq k \leq j \leq n$. The Clifford algebra associated with the pair (V, q) is obtained by demanding that $v \otimes w + w \otimes v = -2q(v, w)$ holds inside $\bigotimes V$ for each $v, w \in V$. Formally this is achieved by taking the quotient of $\bigotimes V$ by the two-sided ideal generated by all elements of the form $v \otimes + w \otimes v + 2q(v, w)$. More precisely, we define the two-sided ideal

$$\begin{aligned} \mathcal{I}_C(V, q) &= \langle v \otimes + w \otimes v + 2q(v, w) : v, w \in V \rangle \\ &= \operatorname{span}\{\alpha \otimes (v \otimes + w \otimes v + 2q(v, w)) \otimes \beta : v, w \in V, \alpha, \beta \in \bigotimes V\} \end{aligned}$$

and then define the ***Clifford algebra*** associated with (V,q) to be the quotient space

$$\mathcal{C}l(V,q) = \bigotimes V / \mathcal{I}_C(V,q)$$

which, of course, inherits an algebra structure from $\bigotimes V$. Now by definition, inside the Clifford algebra, we have

$$v \otimes w + w \otimes v = -2q(v,w), \quad \text{for every } v, w \in V \tag{3}$$

A priori we don't know that the Clifford algebra is uniquely defined, but this will follow immediately from the forthcoming universal property. First observe that V is naturally isomorphic to a subspace of $\bigotimes V$, via the embedding $\mathsf{i} : v \mapsto 0 \oplus v \oplus 0 \cdots$. Then composing with the canonical projection, this embedding descends to a linear map $\mathsf{i} : V \to \mathcal{C}l(V,q)$ satisfying

$$\mathsf{i}(v)^2 = \mathsf{i}(v) \otimes \mathsf{i}(v) = v \otimes v = -q(v,v)$$

which shows in particular that i is injective if and only if q is nondegenerate.

Theorem 1 (Universal Property of Clifford Algebra)**.** *For any unital associative algebra $\mathcal{A}$ over $\mathbb{C}$ and any linear map $\mathsf{j} : V \to A$ satisfying $\mathsf{j}(v)^2 = -q(v,v)1_{\mathcal{A}}$ for every $v \in V$, there exists a unique algebra homomorphism $f : \mathcal{C}l(V,q) \to \mathcal{A}$ such that $f \circ \mathsf{i} = \mathsf{j}$. In other words, j extends uniquely to an algebra homomorphism on $\mathcal{C}l(V,q)$.*

In particular, the universal property immediately implies that $\mathcal{C}l(V,q)$ is uniquely defined up to isomorphism (of $\mathbb{C}$-algebras), so it makes sense to talk about *the* Clifford algebra associated with (V,q).

Now for any Riemannian manifold (M,g), at each $p \in M$ we obtain a bilinear form on the tangent space at p, $g_p : T_pM \times T_pM \to \mathbb{R}$, and therefore we have at each point a real Clifford algebra $\mathcal{C}l(T_pM, g_p)$. However, we will be more interested in the Clifford algebra associated with the cotangent spaces, $\mathcal{C}l(T_p^*M, g_p)$. Then we can consider the vector bundle of Clifford algebras on the cotangent bundle of M; i.e. the disjoint union

$$\mathcal{C}l(T^*M, g) = \bigsqcup_{p \in M} \mathcal{C}l(T_p^*M, g_p)$$

which we might call the ***Clifford bundle*** associated with the pair (T^*M, g), or something like that, but I haven't seen any standard terminology for this object.

More generally, given any vector bundle $E \to M$ equipped with a bundle metric g_E, there is a well-defined vector bundle $\mathcal{C}l(E, g_E)$ whose fiber over each $p \in M$ is the Clifford algebra $\mathcal{C}l(E_p, g_{E_p})$. For our purposes in this note we will not need to go beyond the case $E = T^*M$.

Remark. We recall that, by definition, a Riemannian manifold is a pair (M,g), where M is a smooth manifold, and g is a smooth map on M whose value g_p at any point $p \in M$ is an inner product on T_pM. In other words, g is a smooth, symmetric, positive-definite, covariant 2-tensor field on M. Note that in this context, since each tangent space is an inner product space, we have a canonical isomorphism $T_pM \simeq T_p^*M$ and thereby we also obtain inner products on each cotangent space. In light of this identification, in the preceding paragraph we denoted the inner product on both T_pM and T_p^*M by g_p.

3 Clifford modules

We are not just interested in Clifford algebras, but also in ($\mathbb{C}$-algebra) representations of Clifford algebras; that is, understanding how the Clifford algebra acts on another vector space. More generally, we are interested not only in the bundle of Clifford algebras on a Riemannian manifold, but also in representing that bundle as bundle endomorphisms acting on some other vector bundle. In fact, notice that equation (2) is not exactly analogous to (3) despite the obvious similarity – the former is a relation between linear maps on a vector space, whereas the latter is a relation between vectors themselves. Passing from (2) to (3) is precisely the work of a representation $\mathcal{C}l(V, q) \to \text{End}(W)$ transforming tensor products into compositions. These remarks lead us directly to the notion of a Clifford module.

Let $\mathcal{A}$ be a unital, associative $\mathbb{C}$-algebra. A ***representation*** of $\mathcal{A}$ is a pair (θ, W) where W is a vector space over $\mathbb{C}$ and $\theta : \mathcal{A} \to \text{End}(W)$ is an algebra homomorphism, where we're thinking of $\text{End}(W)$ as a $\mathbb{C}$-algebra whose bilinear product is composition of endomorphisms. More explicitly, we mean that $\theta : \mathcal{A} \to \text{End}(W)$ is a map satisfying:

(i) $\theta(au + bv) = a\theta(u) + b\theta(v)$

(ii) $\theta(u \cdot v) = \theta(u) \circ \theta(v)$

(iii) $\theta(1_{\mathcal{A}}) = \text{Id}_W$

for every $a, b \in \mathbb{C}$ and $u, v \in \mathcal{A}$. Thus, the representation allows us to think of the algebra $\mathcal{A}$ as acting on the vector space W. Now the generalization to vector bundles is just what you would expect: given a vector bundle $E \to M$ whose fiber E_p at each point $p \in M$ is a $\mathbb{C}$-algebra, a ***representation*** for E is a pair (θ, F) where $F \to M$ is another vector bundle over M and $\theta : E \to \text{End}(F)$ is a vector bundle homomorphism which restricts to a representation of $\mathbb{C}$-algebras on each fiber. Explicitly, we mean that for every $p \in M$, the restriction $\theta|_p : E_p \to \text{End}(F_p)$ is a representation of $\mathbb{C}$-algebras.

We are interested in the situation where (M, g) is a Riemannian manifold and $E = \mathcal{C}l(T^*M, g)$ is the bundle of Clifford algebras. Suppose we have a representation $\theta : \mathcal{C}l(T^*M, g) \to \text{End}(F)$, then on each fiber θ must satisfy (suppressing the pointwise notation)

$$\begin{aligned} \theta(v)\theta(w) + \theta(w)\theta(v) &= \theta(v \otimes w + w \otimes v) \\ &= \theta(-2g(v, w)) \\ &= -2g(v, w)\,\text{Id} \end{aligned} \tag{4}$$

and such a representation for $\mathcal{C}l(T^*M, g)$ is called a ***Clifford map*** or ***Clifford multiplication***. The vector bundle F (on which $\mathcal{C}l$ acts) is called a ***Clifford module***. Conversely, if $\theta : E \to \text{End}(F)$ is a representation such that, on each fiber θ satisfies $\theta(v)\theta(w) + \theta(w)\theta(v) = -2g(v, w)\,\text{Id}$, then by Theorem 1, θ extends to a representation $\theta : \mathcal{C}l(T^*M, g) \to \text{End}(F)$.

Notice that (4) is exactly the relation (2) we derived as a necessary condition for the existence of a Dirac operator on Minkowski space $M = \mathbb{R} \times \mathbb{R}^3$ equipped with the Riemannian metric $g(u, v) = u^T \eta v$. Thus, applying our new terminology to the old context, (2) says that a Dirac operator on Minkowski space exists if and only if there is a Clifford multiplication which assigns the standard orthonormal basis in $\mathbb{R}^4$ to the Dirac matrices, $e_j \mapsto \gamma_j$.

So, motivated by the connection of these two ideas, we might propose the following more general hypothesis:

> **Hypothesis.** Given a Riemannian manifold (M, g) and a vector bundle $E \to M$, we can find a Dirac operator if and only if E is a Clifford module with respect to some Clifford multiplication map $\theta : \mathcal{C}l(T^*M, g) \to \mathrm{End}(E)$. (5)

Of course, we need to make some of these details precise: what is a Dirac operator in this more general setting? What is the operator supposed to be acting on? In the next section we will clarify these details and then prove that our hypothesis has an affirmative answer.

4 Dirac operators

A ***differential operator*** on $\mathbb{R}^n$ is just a $C^\infty(\mathbb{R}^n)$-combination of partial derivative operators that acts linearly on smooth functions by differentiation; in other words, it's a linear map $D : C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ that looks like

$$D = \sum_{|\alpha| \leq m} a_\alpha(x) \partial_x^\alpha$$

where each $\alpha = (j_1, \ldots, j_n)$ is a multi-index with $|\alpha| = j_1 + \cdots + j_n$, each a_α is a smooth function of $x \in \mathbb{R}^n$ and $\partial_x^\alpha = \partial_{x_1}^{j_1} \cdots \partial_{x_n}^{j_n}$. In this case the positive integer m is called the ***order*** of D. An example of a second-order constant coefficient differential operator on $\mathbb{R}^n$ is the Laplacian,

$$\Delta = -\sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$$

and a first-order differential operator satisfying $D^2 = D \circ D = \Delta$ is called a ***Dirac operator*** on $\mathbb{R}^n$. Here's a rough outline of how we will proceed to generalize these concepts to the setting of a vector bundle $E \to M$ over a Riemannian manifold:

1. A differential operator acts on sections of a vector bundle $E \to M$, and just looks like a Euclidean differential operator after local trivialization.
2. The principal symbol of a differential operator isolates the highest order terms and replaces partial derivatives with cotangent variables.
3. A differential operator is Laplace-type if its principal symbol at each point is multiplication by a scalar.
4. A differential operator D is Dirac-type if it's symmetric and D^2 is Laplace-type.

Let (M, g) be a Riemannian manifold and let $\pi : E \to M$ be a vector bundle. Let $\Gamma(E)$ denote the sections of E, endowed with a vector space structure via fiber-wise operations. Let $D : \Gamma(E) \to \Gamma(E)$ be a linear map, and suppose that for every $p \in M$ we can find local coordinates $\varphi : U \subseteq M \to \mathbb{R}^n$ and a local trivialization $\phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ at p, such that for any local section $\mu : U \subseteq M \to \pi^{-1}(U)$ we have

$$\begin{aligned} D\mu &= \sum_{|\alpha| \leq m} a_\alpha(p) \partial_x^\alpha (\pi_{\mathbb{R}^k} \circ \phi \circ \mu) \\ &= \sum_{|\alpha| \leq m} a_\alpha(p) \partial_x^\alpha (\widehat{\mu}) \end{aligned}$$

then D is called a (order m) ***differential operator*** on sections of E, or just a ***differential operator*** on E. The main idea here is that, after choosing local coordinates and local trivializations, sections of E just look like smooth maps $\mathbb{R}^n \to \mathbb{R}^k$, and D just acts like a Euclidean differential operator. Following this notation, the ***principal symbol*** of D is the map $\sigma_m(D) : T^*M \to \operatorname{End}(E)$ given by

$$\sigma_m(D)(p, \omega_p) = i^m \sum_{|\alpha| = m} a_\alpha(p) \omega^\alpha$$

where $\omega = \omega_1 dx^1 + \cdots \omega_n dx^n$ in local coordinates and $\omega^\alpha = w_1^{\alpha_1} \cdots \omega_n^{\alpha_n} \in C^\infty(M)$. The main idea here is that we're just taking just top-order term of D and replacing the partial derivatives with cotangent variables. (The factor of i^m is a normalization factor that simplifies calculations.)

Now we have all of the groundwork set to generalize the concepts of Laplace and Dirac operators. A second-order differential operator L on E is ***Laplace-type*** if, at each $p \in M$, its principal symbol is given by

$$\sigma_2(L)(p, \omega_p) = g_p(\omega_p, \omega_p) \operatorname{Id}_{E_p} = |\omega_p|^2 \operatorname{Id}_{E_p}$$

that is, at each $p \in M$ the principal symbol acts on E_p by scalar multiplication by $|\omega_p|^2$. A first-order differential operator D on E is ***Dirac-type*** if $D^2 = D \circ D$ is Laplace-type. It is now straightforward to verify in the affirmative one direction of our hypothesis (5):

Fact 1. *A Dirac-type operator on E induces a Clifford module structure on E.*

Proof. Write $\theta = \sigma_1(D)$. Since D is symmetric and $D^2 = L$ is Laplace-type, we have for any $(p, \omega_p) \in T^*M$,

$$\begin{aligned} \theta(p, \omega_p)^2 &= \sigma_2(D \circ D)(p, \omega_p) \\ &= \sigma_2(L)(p, \omega_p) \\ &= g_p(\omega_p, \omega_p) \operatorname{Id}_{E_p} \end{aligned}$$

Now applying this formula to $\nu + \omega \in T^*M$ we have

$$\begin{aligned}[g(\nu,\nu) + g(\omega,\omega) + 2g(\nu,\omega)]\,\mathrm{Id} &= g(\nu+\omega,\nu+\omega)\,\mathrm{Id} \\ &= \theta(\nu+\omega)^2 \\ &= (\theta(\nu)+\theta(\omega))(\theta(\nu)+\theta(\omega)) \\ &= \theta(\nu)^2 + \theta(\nu)\theta(\omega) + \theta(\omega)\theta(\nu) + \theta(\omega)^2 \\ &= g(\nu,\nu)\,\mathrm{Id} + g(\omega,\omega)\,\mathrm{Id} + \theta(\nu)\theta(\omega) + \theta(\omega)\theta(\nu)\end{aligned}$$

And consequently

$$\theta(\nu)\theta(\omega) + \theta(\omega)\theta(\nu) = 2g(\nu,\omega)\,\mathrm{Id}$$

So by our previous remark, $\theta : T^*M \to \mathrm{End}(E)$ extends to a representation $\theta : \mathcal{C}l(T^*M, g) \to \mathrm{End}(E)$. Hence E is a Clifford module and the Clifford multiplication is given by the principal symbol of D, $\theta = \sigma_1(D)$.

■

Now we will prove the converse of Fact 1.

Fact 2. *A Clifford module admits Dirac-type operators.*

Proof. Conversely, let $\pi : E \to M$ be a Clifford module over M with Clifford multiplication $\theta : \mathcal{C}l(T^*M, g) \to E$. We will construct a Dirac-type operator D on E by first defining an operator locally on an open neighborhood of any point, and then use a partition of unity to piece together the local operators. For any $p \in M$, choose local coordinates $(x_1, \dots, x_n)$ in a locally trivial open neighborhood U_α containing p, and let $\phi : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k$ denote the local trivialization for both $T^*M|_{U_\alpha}$ and $E|_{U_\alpha}$. Define a differential operator $D_\alpha : \Gamma(E|_{U_\alpha}) \to \Gamma(E|_{U_\alpha})$ by

$$D_\alpha(\mu) = \sum_{j=1}^{m} \theta(dx^j)(\widehat{\mu})$$

for any local section $\mu \in \Gamma(E|_{U_\alpha})$, expressed in local coordinates as $\widehat{\mu} = \pi_{\mathbb{R}^k} \circ \phi \circ \mu$. Then, choosing a partitution of unity $\psi_\alpha : U_\alpha \to \mathbb{R}$ subordinate to the open cover (U_α) for M, we define a Dirac-operator on E by $D = \sum_\alpha \psi_\alpha D_\alpha$. ■

In light of these two facts we can promote hypothesis (5) to the status of a theorem:

> **Theorem.** Given a Riemannian manifold (M, g) and a vector bundle $E \to M$, we can find a Dirac operator on E if and only if E is a Clifford module.

5 Open questions

Not "open" questions per se, but just some ideas/questions that occurred to me while writing this note, and which I don't have satisfactory answers to.

1. Does every vector bundle admit a Clifford module structure / Dirac operators? Is there a manifold where there is no Clifford module / Dirac operators?
2. How can we classify all Dirac operators on a vector bundle? Can we construct a one-to-one correspondence between Dirac operators and some other collection of objects?
3. There may be a lot of necessary and sufficient relations or conditions that characterize Dirac operators – what makes the Clifford algebra structure an important or useful one?

References

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